

# Real Options Approaches to Conservation Easements

Paul L. Fackler, Jacob N. Brimlow and Evan Mercer\*

February 5, 2007

## Abstract

Conservation easements are widely used to preserve land from development. Few previous studies have examined landowner decisions concerning the desirability of accepting easements. Here a real options approach is taken to examine this issue. An important distinction is made between easements that are offered on a one-time basis and standing offers. Also examined are effects of property taxes, land use regulations and restrictions and alternative easement payment provisions.

---

\*The authors are Associate Professor in the Department of Agricultural and Resource Economics and Graduate Research Assistant at North Carolina State University, and Research Economist, Southern Research Station, USDA Forest Service.

Correspondence: paul.fackler@ncsu.edu

©2007, Paul L. Fackler

# 1 Introduction

A growing human population and the accompanying development pressure have increased concerns that critical habitat and open space are being lost at socially unacceptable rates. For example, funds totaling \$30.6 billion were approved for conservation uses at the U.S. state, county and municipal level between 1994 and 2005 (Plantinga). Governments and land trusts secure environmental service flows using several methods, including land use regulation, fee simple ownership, and conservation easements or contracts. Land use regulations such as the Endangered Species Act and local zoning ordinances mandate conservation, removing use rights from landowners. Fee simple ownership is the outright purchase of land for conservation, and conservation easements are legal contracts that remove specific use rights, often in perpetuity, from property (Boyd, et al.).

Conservation easements are of particular interest because they are widespread: the percentage of total land trust acreage held in conservation easements grew from 33 to 78 percent from 1984 to 2003, totaling over 5 million acres (Parker), and the 2002 U.S.D.A. Farm Bill authorized enrollment of over 40 million acres in easement and easement-type<sup>1</sup> programs such as the Conservation Reserve, Working Forest Legacy, Wetland Reserve and Wetland Protection programs.

Tax incentives have played an important role in conservation easement transactions. Na-

---

<sup>1</sup>The Conservation Reserve Program enrolls land in 10 to 15 year contracts which can be broken if all payments plus a 25% penalty are repaid. Although the distinction between perpetual and temporary easements is potentially important, only perpetual easements are discussed here.

tional and local tax laws reducing the tax burden on landowners who accept easements have provided incentives for voluntary donation, but these tax effects are typically too small to fully explain donations (Plantinga). Viewed in a slightly broader way, taxes are one of several incentive devices that conservators can use to convince landowners to accept conservation easements. Valuing easements as a means to set compensation levels has proven a difficult hurdle for policymakers (Plantinga). Common methods of valuation are ‘before and after’ or ‘comparable sales.’ Subtracting the encumbered land value from the unencumbered land value yields a before and after valuation of the easement, while comparable easement or land sales can be used to impute value. Difficulties including a backward-looking nature, the necessity for appraisals which are subject to bias and shallow markets have caused skepticism about the accuracy and desirability of these methods (Plantinga). Since conservation easements restrict conservation incompatible uses only, the private cost, and hence the implicit social value, of preservation is revealed and need not be estimated (Boyd, et al).

Conservation easements are traded in quasi-market transactions between private landowners and conservation entities; these transactions can avoid many of the incentive and valuation difficulties of alternative methods. Recent work (Innes, et al., Margolis, et al.) has suggested that land use regulation may induce perverse behavior by placing the cost burden of conservation on private landowners. In a conservation easement transaction, however, a landowner is given a choice, and will accept an easement only if compensated by at least the private opportunity cost of the restriction. By forcing them to pay the cost, conservation easements create incentive for conservators to seek out lands that achieve conservation goals

at minimum cost; this incentive is absent in the extreme case of land use regulation, and confounded by the presence of conservation compatible use values in fee simple purchases. Further, landowners are left to maximize profits over remaining land uses, minimizing costs due to decreased land productivity often found in fee simple transactions (Boyd, et al). Conservation easements can be used to secure flows of environmental services at low upfront costs relative to fee simple ownership, but this benefit must be balanced against the transactions and monitoring costs necessary to secure and enforce easement contracts (Boyd, et al).

In the case where a landowner is approached with a one-time conservation easement offer, Tegene, Weibe, and Kuhn (1999) argue that the value of an easement is the value of the lost option to develop, and the landowner must be compensated in direct payments, tax incentives, or other forms of compensation to be induced to conserve land. The one-time offer scenario supposes that the landowner's only ongoing decision is whether or not to develop. A landowner may, however, face an ongoing decision to accept a known easement offer as well as an ongoing development decision. This 'standing offer' scenario represents many government programs, and leaves the landowner with two options: an option to develop, as above, and an additional option to accept the easement offer. In this paper we examine and compare easement trigger prices in the one-time and standing offer scenarios using the real options framework (Dixit and Pindyck), which incorporates the effects of uncertainty, irreversibility and the value of waiting. We also examine and discuss the policy implications of how landowner decisions are affected by property and other taxes and the possibility of land use regulation.

An understanding of private landowner incentives can aid policymakers and private land conservators in making informed decisions about the effects of the easement programs they design. The distinction between a one-time and a standing offer highlights the importance of a careful consideration of program details. Other program details, including the size and nature of tax incentives, the effect of land use restrictions also matter and can be analyzed in the framework developed here. Additionally, our examination of the behavioral implications of property tax policies and impending land use regulation may offer the policymaker useful warnings about policy pitfalls.

The paper proceeds as follows: Section II reviews the land use and valuation literature, with specific attention paid to the concept of option value and relevant uses of the real options framework. Section III introduces the basic one-time offer model, which involves a stochastically evolving development value as the single source of uncertainty, and provides an explicit expression for the willingness to accept one-time offers of easements. Section IV examines the landowner's decision problem when a standing offer exists and compares the results of the one-time and standing offer models. Section V contains extensions involving taxes, takings, contract provisions (limitations on conservation compatible uses) and easement payment structure. The paper concludes in Section VI with a closing discussion and ideas for future research. A mathematical appendix is attached.

## 2 Literature

The environmental economics literature recognizes several definitions of option value (Haneman, 1989). Developed originally by Cicchetti and Freeman (1971), *option value* has been interpreted as a risk premium due to the uncertain future value of environmental goods. Arrow and Fisher (1974) and Henry (1974) define the concept of *quasi-option value*, a measure that highlights irreversibility and incorporates the possibility that useful information will arrive over time. The *real option* value of Dixit and Pindyck (1994) (DP) is also concerned with irreversibility and uncertainty and is the difference between the value of two otherwise identical assets, one of which incorporates an option. DP also examine how the value of an asset changes when a decision must be taken immediately compared to one for which a decision can be delayed. Fisher (2000) asserts that the DP option value and quasi-option value are equivalent. Mensink and Requate (2004), however, show that the DP value can be broken into two parts: the first, equivalent to the quasi-option value and due to the presence of uncertainty, is the value of obtaining new information; the second is the value of postponement irrespective of uncertainty. Both aspects contribute to the value of a postponement option. We will use the DP option value in our analysis of decisions made by landowners holding postponement options.

Including option values in many irreversible investment decision problems shows that it may be optimal for decision-makers to delay decisions beyond the point at which traditional net present value (NPV) thresholds for investment are met (Henry, 1974, McDonald and Seigel, 1986, Pindyck, 1991). However, Pindyck (2000,2002) shows that the type of uncer-

tainty matters: in some cases, the presence of uncertainty can lead to investment thresholds below those dictated by traditional NPV analysis. Our model generates decision price thresholds that incorporate development price uncertainty and the possibility of postponement, and imply that the landowner will be more likely to wait than if decisions were made in a myopic NPV framework.

The concept of land conversion as an investment opportunity and the importance of including option values in land use valuation are well established. Titman (1985) demonstrates the importance of viewing use rights as irreversible options on real assets in a model of urban land prices that explains vacant lots in downtown Los Angeles. Clark and Reed (1988) find that uncertainty affects the value of vacant land proportionally more than that of developed land due to the remaining option to develop. Quigg (1993) analyzes property values in Seattle, finding empirical support for an ‘option premium’ where delayed development is possible and uncertainty is present. Cunningham (2006) finds similar evidence that uncertainty creates an option premium, finding that real options raise land prices on Seattle’s rural/urban fringe.

The real options framework has also been applied to rural land use decisions. Conrad (1997) examines the value of an old growth forest using a single-factor model in which the future value of timber is known but non-timber amenity values evolve according to a geometric Brownian motion process. Forsyth (2000) uses a mean reverting process to show that the results of Conrad’s model are sensitive to the specification of the stochastic process driving amenity value. Wiemers and Behan (2004) employ a real options model to explain

the observed reluctance of farmers to switch to forestry in Ireland, and Behan, McQuinn, and Roche (2006) use the model to motivate an empirical analysis of the response of Irish farmers to a national farm forestry incentive program. Property value and development decision impacts of impending land use regulation are examined by Riddiough (1997), who finds that credible threats of ‘taking’ lead to a substantial decreases in property value and increase the probability of development.

All the aforementioned studies assume only a single source of uncertainty. Multi-factor real options models have been used to examine broader issues of wilderness and ecosystem preservation when cost and/or benefit uncertainties are present. Conrad (2000) develops a model in which development, extraction of a resource, and conservation interact with uncertainties over both amenity values and project returns. Pindyck (2000, 2002) addresses the problem of timing environmental policy actions when future ecosystem evolution and the costs and benefits of environmental degradation are unknown. Solutions to the general specification of the Conrad (2000) and Pindyck (2000,2002) models require multi-factor dynamic programming techniques.

To our knowledge, only one study exists that uses a real options framework to evaluate conservation easement value when irreversible use options, postponement and uncertainty are present. Tegene, Weibe, and Kuhn (1999) model convertible agricultural land as an investment opportunity and relate the value of the option to convert to the value of a conservation easement. The authors find that conventional easement valuation procedures overlook a potentially significant option value, and demonstrate that it may be optimal to

delay a conversion decision even when the future value of conversion is known. Our analysis extends their work by allowing for standing as well as one-time offers, and by considering the effects of tax rate, structure, the possibility of land use regulation and the effect of various contract provisions.

### **3 Easements as One-time Offers**

We assume that landowners dynamically maximize utility over development and preservation land uses, and that converting to one use precludes use of the other. Landowners are uncertain about future returns to development, and are not obligated to act. For a choice (develop or preserve) to be optimal, the landowner must receive at least the opportunity cost of the decision, i.e., the minimum willingness to accept a conservation easement. This opportunity cost includes the option values that account for uncertainty and the value of waiting.

Consider a landowner that derives a stream of revenue/amenity benefits worth  $A$  per unit of time. If the landowner discounts the future at rate  $r$ , the present value of the land if it remains perpetually in its current use is  $A/r$ . The land could also be sold to a developer for a price  $P$  that is varying randomly over time. We assume that the development price is

described by the following geometric Brownian motion:<sup>2</sup>

$$dP = \mu P dt + \sigma P dW$$

The landowner can either do nothing and collect the revenue/amenity stream  $A$  or sell the land for development for price  $P$ . Once sold the landowner gets no further value from the land and the problem stops. Optimal stopping problems of this type are well studied and have been discussed extensively, for example, in Dixit and Pindyck (1994). The optimal decision rule can be expressed in terms of a partition of  $P$  into a region where it is optimal to wait and one where it is optimal to sell. In the waiting region the value of the land,  $V(P)$ , satisfies the differential equation

$$rV(P) = A + \mu P V_P(P) + \frac{\sigma^2}{2} P^2 V_{PP}(P)$$

subject to boundary conditions.

The solution to this differential equation has the form

$$V(P) = \frac{A}{r} + \alpha_1 P^{\beta_1} + \alpha_2 P^{\beta_2}$$

where the  $\beta_i$  are the roots of the characteristic equation<sup>3</sup>

$$0 = \beta(\beta - 1)\sigma^2/2 + \beta\mu - r.$$

---

<sup>2</sup>Following standard practice in finance, the process describing  $P$  is under the risk-neutral or martingale-equivalent measure rather than the actual probability measure. The two are identical if there is no risk premium attached to the development price. The use of the risk neutral process is essentially equivalent to using certainty equivalent returns and discounting at the risk free rate, thereby avoiding the need to evaluate returns using risk adjusted discount rates.

<sup>3</sup>In the deterministic ( $\sigma = 0$ ) case, when  $\mu > 0$ , there is a single positive root equal to  $r/\mu$ .

and the  $\alpha_i$  are constants to be determined. For economically relevant parameter values ( $r > 0$  and  $r > \mu$ )  $\beta_1 < 0$  and  $\beta_2 > 1$ .

The optimal decision is to sell the land for development whenever  $P$  exceeds some cutoff price  $P^*$ . Zero is an absorbing barrier of the price process, so a price of zero implies that development will never occur. The constant,  $\alpha_1$ , associated with the negative root must therefore be zero to ensure the intuitively reasonable requirement that  $V(0) = A/r$ .  $P^*$  and the constant  $\alpha_2$  are chosen to satisfy a value matching condition  $V(P^*) = P^*$  and a smooth-pasting condition that  $V'(P^*) = 1$ , yielding the result that

$$P^* = \frac{\beta_2}{\beta_2 - 1} \frac{A}{r}$$

and

$$\alpha_2 = \frac{1}{\beta_2} (P^*)^{1-\beta_2}$$

Given that  $\beta_2 > 1$ ,  $P^*$  exceeds the current use value of the land  $A/r$ . The value of the land can thus be expressed as

$$V(P) = \begin{cases} \frac{A}{r} + \alpha_2 P^{\beta_2} & \text{for } P < P^* \\ P & \text{for } P \geq P^* \end{cases}$$

The optimal cutoff price  $P^*$  decreases with the discount rate  $r$  and increases with the mean rate of price increase,  $\mu$ , the price volatility,  $\sigma$ , and the amenity value,  $A$  (see the appendix for details of this derivation). When the discount rate is high, i.e., when the opportunity cost of waiting is high, landowners will sell at a lower price. On the other hand landowners require a higher selling price when there is strong upward movement in the

expected development value or when price uncertainty is high. It should be noted that this does not imply that the time until development is longer for larger values of  $\mu$  and  $\sigma$ ; the expected time until  $P$  reaches any given point declines with increases in both  $\mu$  and  $\sigma$ .

The model thus far has no provision for easements. Suppose the landowner is approached with a one time offer of  $E$  to sell the development rights. The value  $E$  could include direct payments from either private or governmental agencies, the amortized value of future tax credits or the monetary value of non-pecuniary benefits to the landowner stemming from personal values for conservation.<sup>4</sup> Under an easement the landowner continues to receive the revenue/amenity flow  $A$  in perpetuity, but gives up the right to sell the land for development. The offer will be accepted if  $V(P) < A/r + E$ , i.e., if the current value of the land, including the development option, is worth less than the current use value plus the easement payment. The landowner decision can be summarized as

$$\begin{aligned} \text{develop} & \quad \text{if } P > \max(A/r + E, P^*) \\ \text{wait} & \quad \text{if } E < \alpha_2 P^{\beta_2} \text{ and } P < P^* \\ \text{accept} & \quad \text{otherwise} \end{aligned}$$

The solution is illustrated in Figure 1 for the following parameter values:  $\mu = 0.02$ ,  $\sigma = 0.1$ ,  $r = 0.05$  and  $A = 1$ . The first three parameter are all expressed in annual rates. Setting  $A$  to equal to 1 normalizes monetary values relative to the size of the annual revenue/amenity flows. With these parameters, one obtains  $P^* = 40$ ,  $A/r = 20$ ,  $\alpha = 1/80$  and  $\beta_2 = 2$ , so

---

<sup>4</sup>Plantinga (2004) argued that there are no financial reasons for setting up conservation easements (in the absence of direct payments). Observed behavior, however, suggests that some landowners are willing to forego development rights even in the absence of such payments.

$V(P) = 20 + P^2/80$  for  $P < 40$  and  $V(P) = P$  for  $P \geq 40$ . Thus the current use value of the land is 20 times the annual return and the development cutoff price is twice the current use value of the land.

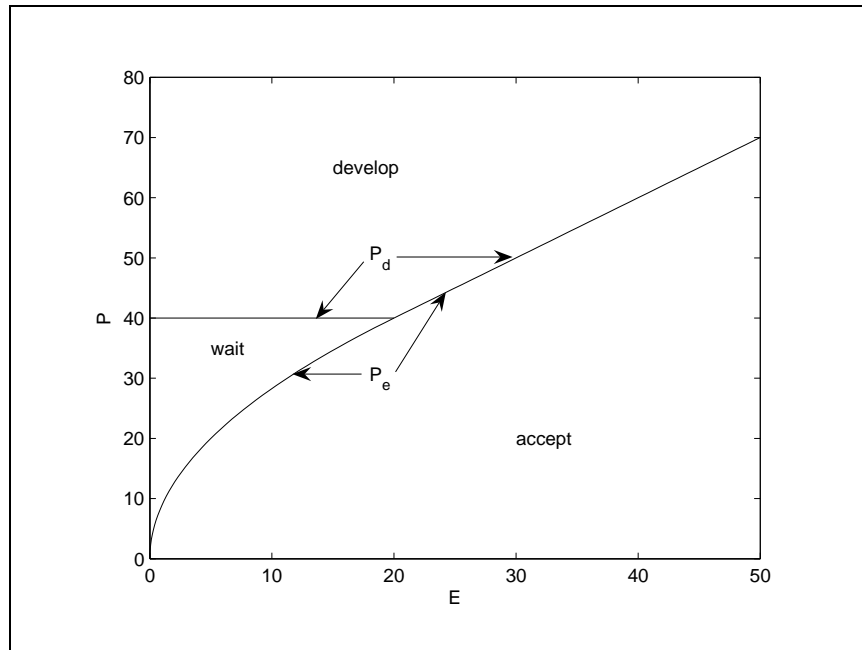


Figure 1. Decision Rule for One-time Offer Model

Figure 1 shows that the  $(E, P)$  space can be partitioned into three regions. At prices below  $P_e$  it is optimal to accept the easement offer. At prices above  $P_d$  it is optimal to develop. For prices above  $P^*$  the  $P_e$  and  $P_d$  curves coincide. For lower prices, however, they diverge, leaving a waiting region between the two curves.

By accepting the easement, the landowner is choosing to forgo development as well as to forfeit the option to develop in the future. Therefore, in the waiting region where the development value  $P$  is below the critical value  $P^*$ , the opportunity cost of conservation is represented by the value of the option to develop,  $\alpha_2 P^{\beta_2}$ , and payments necessary to induce

the acceptance of easements will rise or fall with the value of this option. This value, and therefore the easement payment required, is increasing in both the average rate of increase  $\mu$  and the volatility of the development value process  $\sigma$  (see appendix for derivation). Thus for land which is subject to strong upward development pressure or volatile development prices, easement offers will have to be relatively high to induce acceptance. The effect of changes in the discount rate,  $r$ , are ambiguous however. At low development prices, an increase in  $r$  can lower the option value. At higher values, however, and especially those near  $P^*$ , an increase in  $r$  is associated with a higher option value, implying that higher easement offers are required to induce acceptance. The value of the option decreases with higher amenity/revenue flow  $A$ , reflecting the fact that higher returns in the waiting and easement regions increase the opportunity cost of development.

Since the critical development value  $P^*$  represents the value at which development will occur immediately if there is no offer for easement, it may seem that landowner behavior above this point is trivial and need not be explored. However, it is possible to imagine land being passed down from a landowner who was unwilling to develop at the current price to one who has no such aversion, so that the problem would begin when  $P > P^*$ . Above  $P^*$ , the threshold value is  $E + A/r$ . Increases in the discount rate increase the necessary easement payment as they decrease the value of the amenity/revenue perpetuity ( $A/r$ ). As seen above, higher amenity/revenue flow increases the attractiveness of easement relative to development, decreasing the payment required to induce conservation.

## 4 Standing Offers

The approach taken in the previous section assumes that, from the landowner's perspective, the easement offer is only made once and will never be made again. Suppose, instead, that the offer of an easement is a standing offer. Standing offers represent many United States government conservation programs and are applicable to the forestry payment program in Ireland studied by Weimers and Behan (2004) and Behan, McQuinn, and Roche (2006).

Intuitively, the offer is accepted if  $E$  is high enough relative to  $P$ . Combined with the fact that the landowner will sell if the price is high enough, the decision problem can be viewed as a two-sided optimal stopping problem. The optimal decision is characterized by a value  $P_E$  below which it is optimal to accept the easement and a value  $P_D$ , above which it is optimal to sell the land for development. For values of  $P$  in  $(P_E, P_D)$  it is optimal to wait.

In the waiting region, the same differential equation as in the one time offer case must be satisfied, but with different boundary conditions. The value of the land has the form

$$V(P) = \frac{A}{r} + \eta_1 P^{\beta_1} + \eta_2 P^{\beta_2}$$

where  $\eta_1$  and  $\eta_2$  are constants to be determined. Below  $P_E$ , the value function,  $V(P)$ , equals  $A/r$ , and above  $P_D$  it equals  $P$ . The value matching conditions are therefore

$$A/r + \eta_1 P_E^{\beta_1} + \eta_2 P_E^{\beta_2} = A/r + E$$

and

$$A/r + \eta_1 P_D^{\beta_1} + \eta_2 P_D^{\beta_2} = P_D$$

The first condition states that when  $P = P_E$  the landowner is indifferent between owning the land with both options still alive and owning the land and exercising the easement option, which pays  $E$  for giving up the development option. The second states that when  $P = P_D$  the landowner is indifferent between owning the land with both options still alive and selling the land for price  $P_D$ . The optimal cutoff prices are determined using the smooth-pasting conditions

$$\beta_1 \eta_1 P_E^{\beta_1 - 1} + \beta_2 \eta_2 P_E^{\beta_2 - 1} = 0$$

and

$$\beta_1 \eta_1 P_D^{\beta_1 - 1} + \beta_2 \eta_2 P_D^{\beta_2 - 1} = 1$$

In general, the one-time offer cutoff curves lie between the acceptance and the develop curves for the standing offer. This is true because, like the option to develop, the option to accept a standing easement offer always has a nonnegative value (the holder of an option can choose not to exercise the option if the payoff is negative), and decisions to develop or ease in the standing offer model involve forfeiture of both the option to develop and the option to ease. Therefore, it will not be optimal to develop immediately when a standing easement offer exists unless it is also optimal to develop given a one time offer. Similarly, it is never optimal to accept a standing easement offer unless it is also optimal to accept a one time offer.

The value of use options is positively related to uncertainty; the one-time and standing offer decisions are identical if there is no uncertainty (i.e.,  $\sigma = 0$ ). Furthermore, the spread

between the acceptance and development cutoff curves widens as  $\sigma$  increases. This highlights the importance of development value uncertainty in assessing the impact of conservation easement programs. The two problems are also identical when  $E = 0$  because in this case it is never optimal to enter into an easement.

In general, two sided optimal stopping problems cannot be solved in closed form. An appendix discusses the solution conditions and their numerical solution. To illustrate the nature of the solution, the model is solved using the same parameter values as in the previous section ( $\mu = 0.02$ ,  $\sigma = 0.1$ ,  $r = 0.05$  and  $A = 1$ ). Recall that for the one-time offer model, the cut-off price at which development occurs is  $P^* = 40$ . The standing offer model was solved for values of  $E$  ranging from zero to 50. Figure 2 shows the cutoff prices for the standing offer model. For comparison, the black (middle) curves shows the analogous one-time offer switching curves. The blue (lower) curve represents the price,  $P_E$ , that would cause a landowner to accept a standing offer of  $E$  for an easement. The red (upper) curve represents the price,  $P_D$ , at which a landowner would sell the land for development given various easement payments.

The example illustrates an important aspect of the landowner decision. For a given development price, one-time offers will result in acceptances at lower cost than standing offers. For example, if the current land price is 35, a landowner would only accept a standing offer of 20 or higher but would accept a one time offer of 15.31 or higher. Put another way, a standing offer of  $E = 20$  would be accepted if the current price is at or below 35, but a one-time offer would be accepted if the price is below 40.

This distinction has important implications for easement policy. Where development prices are believed to be significantly below cutoff thresholds and not expected to reach those thresholds in the near future, a one time easement offer program can produce more conservation for a given number of program dollars than a standing offer program. Furthermore, in areas where the development price is near the development cutoff and conservation funds are readily available, a program of one-time offers could yield more current acceptances than a standing offer program, as landowner trigger values are lower than if a standing offer is announced.

If easement offers are high enough, however, a standing offer program can significantly raise the development cutoff price and thereby postpone development.<sup>5</sup> This could be important if current funds for purchasing easements are limited and it is expected that future funds may be more readily available.

Figures 2-4 illustrate the effect of changes in parameter values on the switch points for the standing offer model. Figure 3 shows that  $P_E$  decreases and  $P_D$  increases as the price volatility  $\sigma$  increases, expanding the waiting region. When the possible upside of the development option is high, waiting is more attractive; landowners require higher compensation to forfeit this option through either easement or development.

Figure 4 shows that a similar result holds for the expected rate of price increase  $\mu$ . In this case, the effect is much more pronounced for  $P_D$ : known growth in development value causes the landowner to wait until  $P$  has reached a higher level before development, but

---

<sup>5</sup>The near horizontal behavior of the standing offer development curve at low values of  $E$  indicates (for this example at least) that substantial offers need be made to obtain this postponement effect.

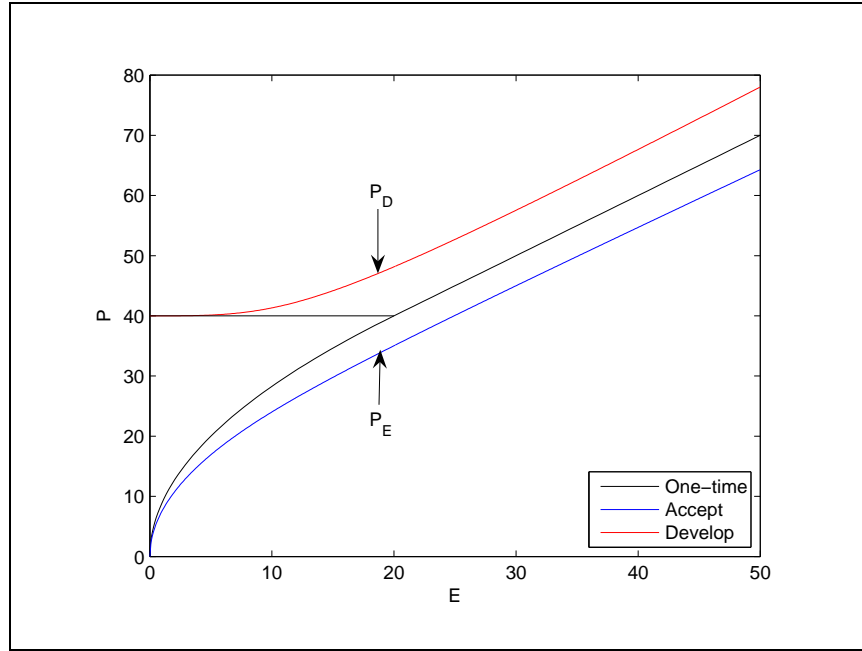


Figure 2. Cut-off Prices for One-time and Standing Offer Models

does not increase  $P_E$  to nearly the same extent.

Increases in the discount rate, illustrated in Figure 5, on the other hand, cause  $P_D$  to drop, as expected. The effect on  $P_E$ , however, is ambiguous and depends on the size of  $E$ .

It would be useful to have a sense of the likelihood of development and the expected time when a decision to either accept an easement offer or develop will be made. For the one-time model, development will occur eventually with probability one if no easement offer is ever made. For the standing offer, the probability that the development price is reached before the easement price, given that the current price is  $P$ , can be shown to equal

$$\text{Prob}(\text{development}) = \frac{P_E^{\beta_1} P^{\beta_2} - P_E^{\beta_2} P^{\beta_1}}{P_E^{\beta_1} P_D^{\beta_2} - P_E^{\beta_2} P_D^{\beta_1}}$$

This probability is shown in Figure 6 for three values of  $E$  (with other parameters set at

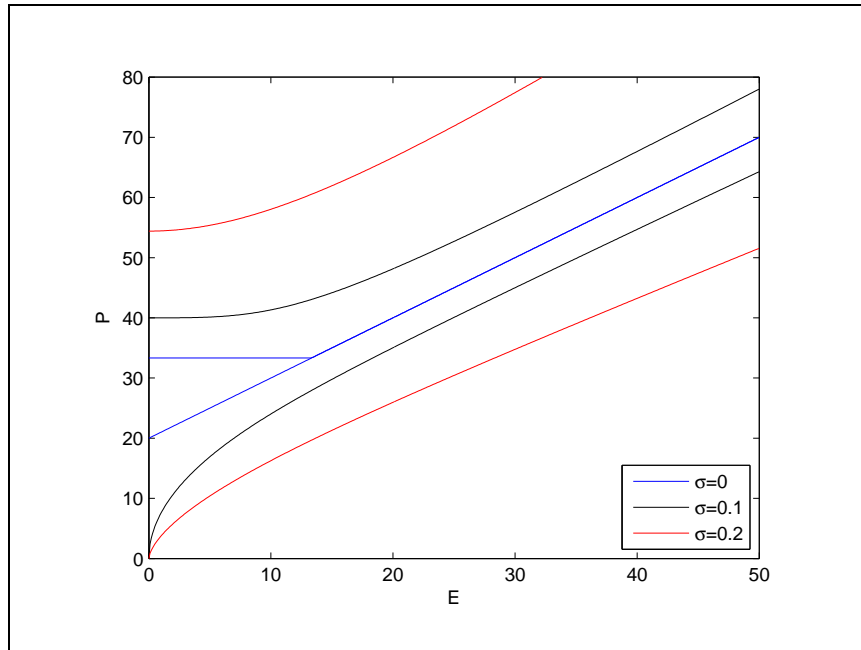


Figure 3. Effect of Price Volatility on Cut-off Prices

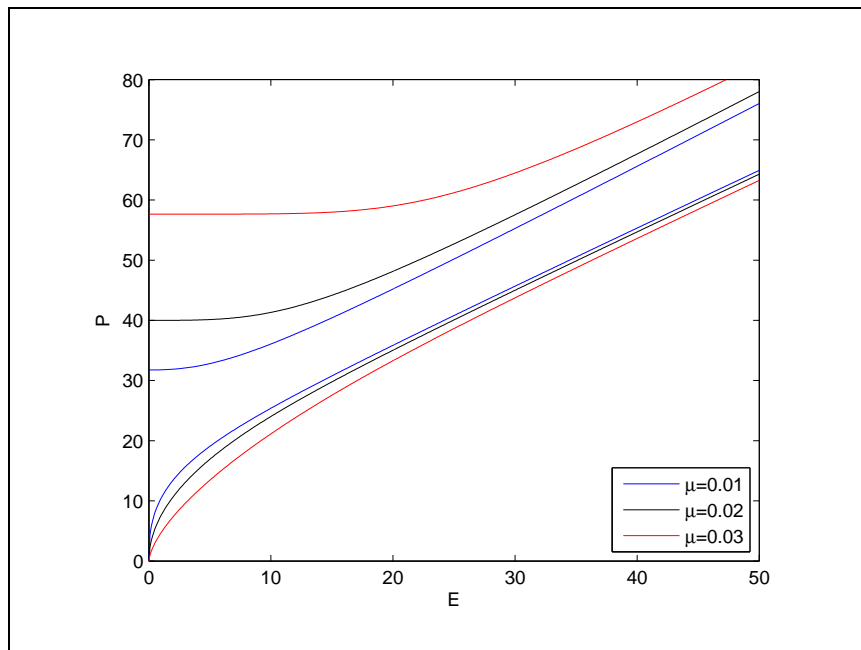


Figure 4. Effect of Mean Price Appreciation Rate on Cut-off Prices

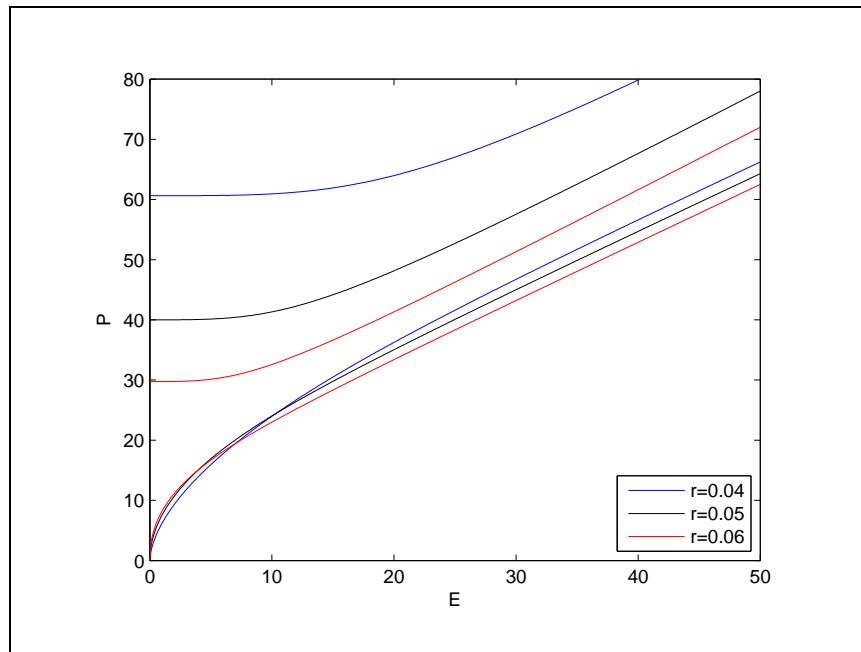


Figure 5. Effect of Discount Rate on Cut-off Prices

their base case values).<sup>6</sup>

For the one-time offer model, the expected time until  $P^*$  is reached (given that current

---

<sup>6</sup>It should be pointed out that the probability values and expected hitting times are evaluated under the same probability measure as is used to compute the optimal decision rule. This is not appropriate if there is a risk premium attached to the development price risk. If so, the actual and risk neutral probabilities will diverge and one should use the actual probability measure to compute hitting time probabilities and expectations (see footnote 2, p. 9).

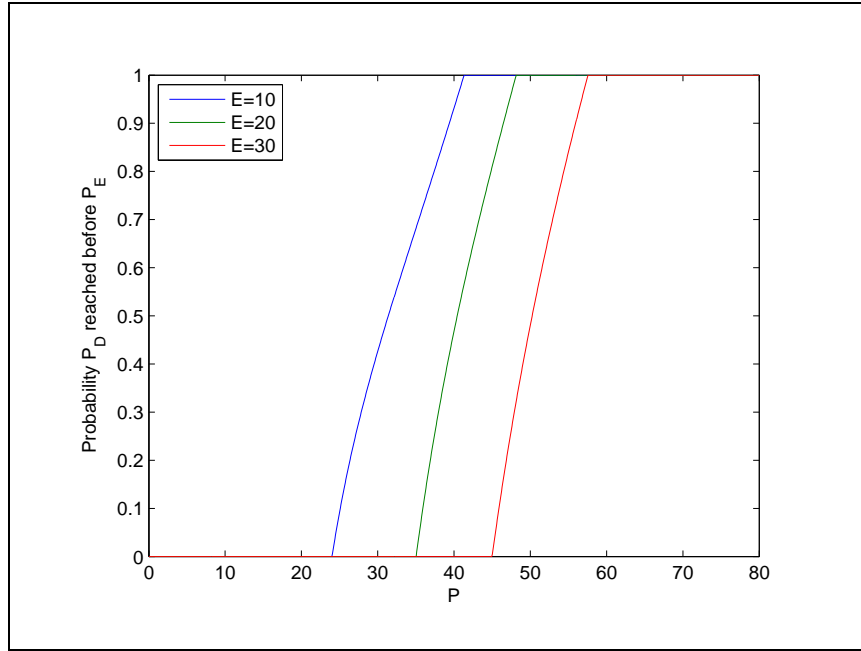


Figure 6. Probability of Development

$P \leq P^*$ ) is<sup>7</sup>

$$\tau(P) = E[\text{time until development}] = \frac{\ln(P^*/P)}{\mu - \frac{\sigma^2}{2}}$$

if  $\mu > \sigma^2/2$  and is infinite otherwise. For the standing offer, the expected time until an action is taken (given that current  $P$  satisfies  $P_E \leq P \leq P_D$ ) is<sup>8</sup>

$$\tau(P) = E[\text{time until action}] = \frac{1}{\mu - \frac{\sigma^2}{2}} \left( \frac{1 - \left(\frac{P}{P_E}\right)^{1-2\mu/\sigma^2}}{1 - \left(\frac{P_D}{P_E}\right)^{1-2\mu/\sigma^2}} \ln\left(\frac{P_D}{P_E}\right) - \ln\left(\frac{P}{P_E}\right) \right)$$

<sup>7</sup>The expected time until hitting satisfies,  $\tau$ ,

$$0 = 1 + \mu P \tau'(P) + \frac{\sigma^2}{2} P^2 \tau''(P)$$

with appropriate boundary conditions (Karlin and Taylor, p.193).

<sup>8</sup>When  $\mu = \sigma^2/2$  the limiting value is  $\ln(P/P_E) \ln(P_D/P)/\sigma^2$ .

The expected time until an action is taken is shown in Figure 7 for the base case parameters, with a standing offer of  $E = 20$ . For the one-time offer case, the development cutoff price is  $P^* = 40$ , whereas for the standing offer case the bounds on the waiting region are 35.03 and 48.15. In the latter case, the expected time until an action occurs is under two and a half years, regardless of the current development price. For the one-time offer case, however, the expected time until development is very high even for current prices moderately less than  $P^*$ .

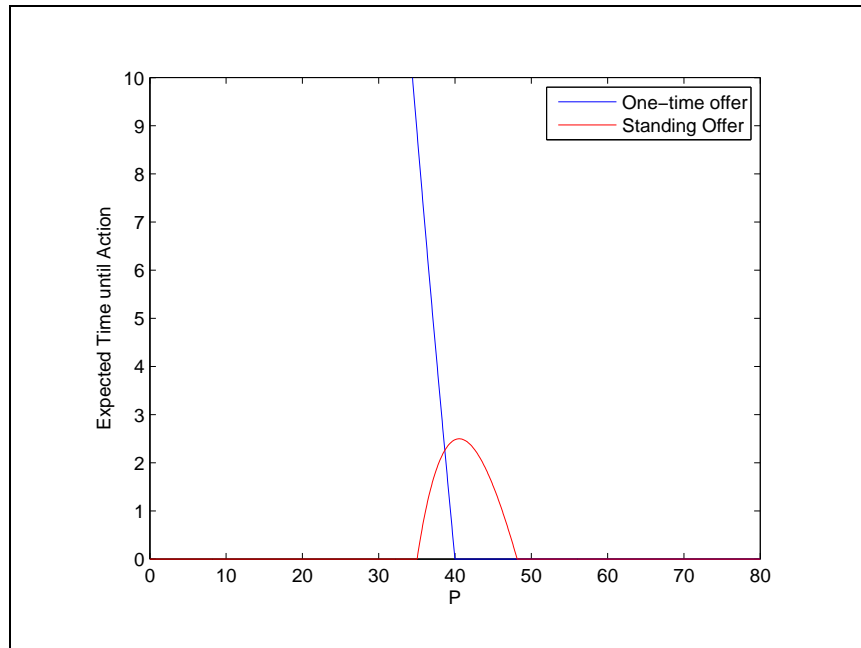


Figure 7. Expected Time Until an Action Occurs

## 5 Extensions

The basic model presented thus far can be extended in various ways to add more realism to the model and to assess the effect of alternative policies. Many of these extensions do not change the difficulty in obtaining solutions but increase the number of adjustment parameters in the model. In this section we include discussions of four extensions: tax effects, land use regulations that affect the possibility of development, easement contract provisions that affect land uses, and easement payments that depend on the current development value of the land. For the sake of brevity, the discussion is mostly limited to the modifications that must be made to the standing offer model but the one-time offer model modifications are completely analogous.

### 5.1 Taxes

Tax incentives are commonly used as compensation to landowners for granting conservation easements, and are consequently thought to be of critical importance in the landowner's decision to accept an easement offer.<sup>9</sup> The present value of future income or estate tax credits or liabilities can be thought of as affecting the one-time value of an easement offer  $E$ . For example, a decrease in estate tax liability for future generations due to the granting of an easement would increase the offer  $E$  by the present value of the tax decrease. The effects

---

<sup>9</sup>Many easement transactions are referred to as 'donations,' but easement donations are most often accompanied by tax compensation of some sort, which suggests that these donations are in fact transactions in which a landowner accepts an easement offer in the form of tax compensation.

of such a decrease in  $E$  are as seen in the one-time and standing offer models above.

Property taxes affect the model in a different way. Suppose that a tax rate of  $\theta$  is assessed on the value of the land, implying that the after-tax amenity flow is  $A - \theta V(P)$ . In this case the value of the land in the waiting region satisfies the differential equation

$$rV(P) = A - \theta V(P) + \mu PV_P(P) + \frac{\sigma^2}{2} P^2 V_{PP}(P).$$

Furthermore, the current use value of the land after an easement is in place satisfies  $V = (A - \theta V)/r = A/(r + \theta)$ . A property tax rate of  $\theta$  is therefore equivalent to changing the required rate of return from  $r$  to  $r + \theta$ . The effects of changes in  $r$  on the one-time and standing offer models were seen above; in both cases, changes in  $r$  significantly affect the development value ( $P_d$  or  $P_D$ ) required to induce development but have a smaller and ambiguous effect on  $P_e$  and  $P_E$ .

One can also consider the possibility that the property tax rate is lower if an easement is accepted. Suppose that after acceptance the tax rate falls from  $\theta$  to  $\theta_E$ . The value matching condition for  $P_E$  becomes

$$\frac{A}{r} + \eta_1 P_E^{\beta_1} + \eta_2 P_E^{\beta_2} = \frac{A}{r + \theta_E} + E$$

where the  $\beta_i$  solve  $0 = \sigma^2/2\beta(\beta - 1) + \mu\beta - (r + \theta)$ . Relative to the full tax treatment, this is equivalent to increasing the easement payment from  $E$  to  $E + \left(\frac{1}{r + \theta_E} - \frac{1}{r + \theta}\right) A$ .

The model presented here cannot, however, be used to assess the impact of property tax changes on development values. To make an assessment of such effects one would have to model how a given tax change would affect the level of  $P$  and/or its dynamics. Thus the

extent to which property tax changes would affect the probability of development or the cost of conservation programs cannot be addressed in this simple model.

## 5.2 Land Use Regulation

The model can also be extended as in Riddiough(1997) to address the issue of land use regulation. Land may be prevented from being developed due to federal policies such as the Endangered Species Act or due to local zoning regulations. Regulation can be modeled as a random one-time elimination of the development right, with the randomness described by a Poisson process with intensity  $\lambda$ . This implies that the probability that the development right will be taken within  $\Delta$  periods is  $1 - \exp(-\lambda\Delta)$ .

In the one-time offer case, the arrival of regulation removes the possibility of development. Since conservation has been mandated, it is reasonable to assume that regulation makes any easement offers moot. For the standing offer model it eliminates both the right to develop and the right to collect  $E$  and enter an easement program. The post-arrival value of the land is therefore  $A/r$  and the differential equation defining the land value becomes

$$rV(P) = A + \mu PV_P(P) + \frac{\sigma^2}{2} PV_{PP}(P) + \lambda(A/r - V(P)).$$

The last term represents the instantaneous probability that a taking occurs times the change in the land value if this event occurs.

The differential equation for the one-time offer model with regulation can be written

$$(r + \lambda)V(P) = (1 + \lambda/r)A + \mu PV_P(P) + \frac{\sigma^2}{2} P_{PP}^V(P).$$

Notice also that the “present value” of the revenue/amenity flow is  $(1 + \lambda/r)A/(r + \lambda) = A/r$ ; hence the only change in the solution is in the values of  $\beta_1$  and  $\beta_2$ , which now solve

$$0 = \beta(\beta - 1)\sigma^2/2 + \beta\mu - (r + \lambda).$$

Viewing the possibility of a regulatory taking as an increase in both the discount rate and the revenue/amenity flow values suggests that this possibility has two effects. First, it makes future flows less valuable, thereby increasing the probability of earlier development. Second, it increases the value of the flows that accrue in the waiting period, decreasing the probability of earlier development.

For the one-time offer model it can be shown (see appendix) that on net an increase in  $\lambda$  lowers the cutoff price,  $P^*$ , leading to development at lower prices. As would be expected, a landowner who expects regulation to eliminate the development rights without compensation will be willing to develop at a lower price. Figure 8 shows how  $\lambda$  affects the optimal cut-off prices for the standing offer model with the base case parameters. Specifically, increasing the probability of regulation that eliminates development rights shrinks the waiting region.

This setup represents the extreme case where taking is complete; the landowner receives no compensation. Less dramatic cases in which a landowner is partially reimbursed for lost value are examined in Riddiough(1997), with the intuitive result that the effects on behavior are in the same direction but less severe. Specifically, if the landowner is compensated at rate  $\rho$  times the development option value  $(V(P) - A/r)$  then the after-arrival value of the land is  $A/r + \rho(V(P) - A/r)$ . This changes the effective intensity rate to  $\lambda(1 - \rho)$ .

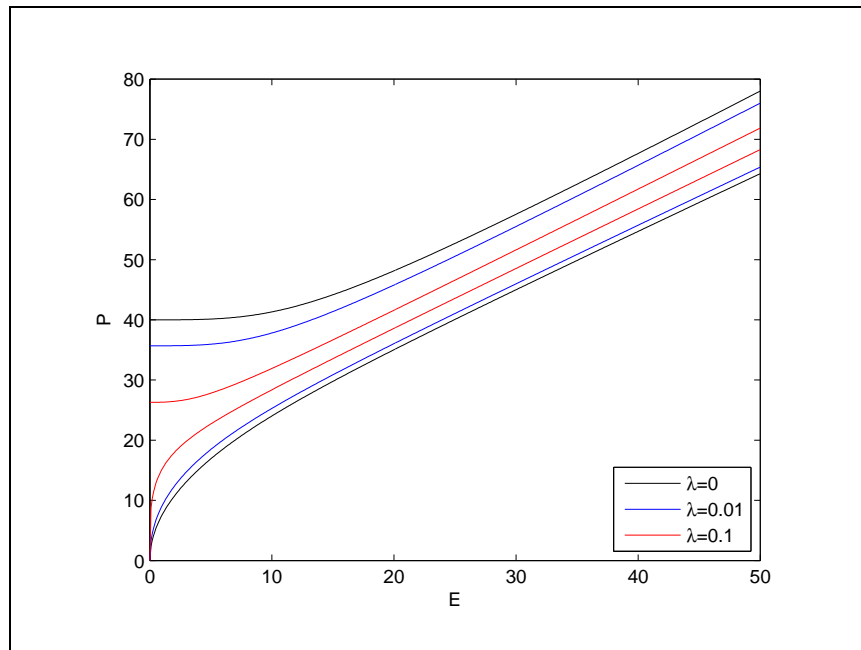


Figure 8. Effect of Takings Probability on Cut-off Prices

### 5.3 Restrictions in Contract Provisions

Thus far our analysis has assumed that a landowner entering an easement agreement will continue to receive the full revenue/amenity flow  $A$  in perpetuity. This could be, among other things, agricultural or forestry profit or environmental amenity flow to the landowner. It is possible, however, that the easement contract may somewhat restrict even conservation compatible uses, suggesting that the flow of revenue/amenity benefits may decrease once the easement is in place. For example, adjacency and other constraints restrict cutting in forestry (Malchow-Moller, et al.) and restrictions on pesticide use and tilling practices can restrict yields in agriculture. The impact of contract provisions can be examined in our framework by taking the post acceptance value of the land to be  $\omega A/r$ , where  $\omega$  is the proportion of revenue/amenity benefit still available under the easement contract. The value matching

condition at  $P_E$  therefore becomes

$$\frac{A}{r} + \eta_1 P_E^{\beta_1} + \eta_2 P_E^{\beta_2} = \omega \frac{A}{r} + E$$

This is equivalent to decreasing the easement payment from  $E$  to  $E - (1 - \omega) \frac{A}{r}$ .

## 5.4 Easement Payments

As discussed above, offers for easements come in a number of forms, including cash payments and tax considerations. In some cases, however, there may be partial compensation for the foregone development option. Suppose that the program offers a fixed payment  $\bar{E}$  plus a fraction  $\psi$  of the forfeited option value so the total offer is  $E = \bar{E} + \psi(V(P) - A/r)$ . The value matching condition and smooth pasting conditions at  $P_E$  in this case are

$$\frac{A}{r} + \eta_1 P_E^{\beta_1} + \eta_2 P_E^{\beta_2} = \frac{A}{r} + \bar{E} + \psi \left( \eta_1 P_E^{\beta_1} + \eta_2 P_E^{\beta_2} \right)$$

and

$$\eta_1 \beta_1 P_E^{\beta_1 - 1} + \eta_2 \beta_2 P_E^{\beta_2 - 1} = \psi \left( \eta_1 \beta_1 P_E^{\beta_1 - 1} + \eta_2 \beta_2 P_E^{\beta_2 - 1} \right)$$

It is easy to see that this has no effect on the smooth pasting condition and that the value matching condition can be rewritten as

$$\eta_1 P_E^{\beta_1} + \eta_2 P_E^{\beta_2} = \bar{E} / (1 - \psi)$$

Thus an easement payment with partial reimbursement for the foregone development option yields a decision rule that is equivalent to using a payment amount of  $\bar{E} / (1 - \psi)$ .

## 6 Conclusions and Future Work

Conservation easements transactions typically occur between private landowners and public or private entities acting on behalf of a public interest. The landowner retains ownership of the property but is asked to restrict use to ensure a flow of environmental services from the land. To be willing to accept the restriction in the form of an easement, the landowner must be compensated by the private opportunity cost; this opportunity cost includes option values that account for uncertainty and the value of waiting. We have examined landowner conservation and development decisions using a one factor real options model in which the development value follows a geometric Brownian motion process. Development and easement price thresholds, along with the probability of development and expected time to development were analyzed for one-time and standing offer scenarios. The behavioral effects of taxes and impending land use regulation were also examined.

Many easement offers can be treated by landowners as one-time occurrences; examples include local land trust attempts to protect environmental services on urban fringe land through individually tailored easement offers. As recognized by Tegene, Weibe, and Kuhn (1999), when landowners view offers in this way, the payment necessary to make a landowner indifferent between accepting the easement and continuing to hold the development option is equal to the value of the development option itself. Landowner's would choose not to develop until the development value  $P$  exceeds some critical value  $P^*$ . Above this critical price, development will occur unless a sufficiently attractive easement offer is made. The easement payment required is increasing in both the average rate of increase  $\mu$  and the

volatility  $\sigma$  of the development price. Therefore, higher easement payments are necessary to induce acceptance from landowners expecting rapid growth of development value. Also high volatility leads to high upside potential, which also increases the payment necessary to induce conservation. The effect of changes in the discount rate depend on the value of  $P$ , but a higher revenue/amenity flow  $A$  decreases the value of the option to develop. A higher value of  $A$  makes both the waiting and easement regions more profitable, thereby decreasing the value of waiting to develop.

Results from the standing offer case are qualitatively similar to that of the one-time offer, but the added option tends to expand the waiting region. For landowners faced with known standing offers, higher values of either the expected rate of increase  $\mu$  or the volatility  $\sigma$  of the development price are associated with a higher development value threshold  $P_D$ . As in the one-time offer case, higher expected development value increases and/or higher value upside potential cause landowners to require a higher development value before committing to development. A lower discount rate, on the other hand, makes waiting less expensive and increases the development value threshold as well.

In the standing offer case, the easement value threshold  $P_E$  describes the price below which an easement with a given payment  $E$  will be accepted. This threshold is generally less responsive than the development threshold to parameter fluctuation; it declines with  $\mu$  and  $\sigma$  but moves ambiguously with the discount rate. The overall effect of increases in the development value parameters and/or decreases in the discount rate is an expansion of the waiting region. This indicates that landowners require higher easement offers  $E$  to be

induced to ease, and higher development value  $P$  to develop.

The distinction between one-time and standing offers for conservation easements proves to be an important one. By introducing a standing offer, each stopping decision (develop, ease) now kills two options: the option to ease and the option to develop. When the development value is below the critical threshold  $P^*$ , this difference yields two policy conclusions: first, a one-time offer program may induce more acceptances at lower easement offers than a standing offer program; second, the introduction of a standing offer program may induce landowners to wait. This has implications for the funding of easement programs; if funding is readily available, a one-time offer program can be used to achieve more conservation per dollar. If, however, funding will increase over time and development pressure is high, introduction of a standing offer program can delay landowner decisions, buying time for policymakers and conservation organizations.

The simple framework we have presented is suitable for a wide variety of extensions. We have discussed the effect of income tax incentives, property tax consequences, land regulations and contract provisions that restrict land uses. All of these extensions are easily accommodated into the basic framework. For example, contract provisions that reduce the amenity/revenue flow to the landowner once an easement is accepted are equivalent to reducing the size of the easement payments,  $E$ , whereas lowering property tax rates on land under easement is equivalent to raising  $E$ .

As previously discussed by Riddiough (1997), impending land use regulation is shown to decrease the development value threshold, implying that policies that increase landowner's

expectation of land use regulation promote development. This effect is widely recognized in regard to the Endangered Species Act (Brown and Shogren, 1998). It suggests that local officials announcing zoning changes before they are enforced may find the land they hoped to protect compromised before laws take effect.

We recognize several avenues for future research. The analysis above, along with Tegene, Wiebe and Kuhn (1999), discusses public and private conservation interchangeably, and does not account for the possibility that the two types of conservation may interact. Theories suggesting that government spending may ‘crowd out’ private charitable giving have been offered by Warr (1982) and Roberts (1984), among others, and the public finance literature contains many empirical studies suggesting that the effect is observed (Abrams and Schmitz 1978, Kigma 1989, Andreoni 1993, Andreoni and Payne 2003). If enrolling in a conservation easement program is a type of charitable giving, the literature suggests that government programs which provide for public goods may impact the level of non-profit activity in the same area. For example, Parker and Thurman (2005) finds empirical evidence that CRP activity partially crowds out land trust conservation purchases. Since many government programs are described by the standing offer model, while much private conservation is achieved through one time offers, a fruitful area for future work would be to examine crowding out by comparing and contrasting the solutions obtained in each of the models.

Our use of Brownian motion is useful as a starting point for describing the process driving development value, and sensitivity analysis shows that the development value rate of increase  $\mu$  and volatility  $\sigma$  play nontrivial roles. Forsyth (2000) and Insley (2002) have shown that the

processes used to describe stochastic variables have important effects on land-use decisions, suggesting that processes other than geometric Brownian motion should be explored in future work. Using the model to inform real world decision making would require estimating the stochastic processes using observed data.

The single factor model could be expanded to include additional stochastic factors and/or more involved descriptions of deterministic variables. Allowing the amenity value to follow a stochastic process would allow analysis of the uncertainty facing landowners regarding the costs and benefits of conservation. In the standing offer case, easement payments could be described stochastically as well, reflecting uncertainty facing landowners regarding the future policy actions of the government. Easement payments and tax structures could also be more specifically described to reflect specific cases.

## References

- Arrow, K., Fisher, A., 1974, "Environmental Preservation, Uncertainty, and Irreversibility." *The Quarterly Journal of Economics*, 88(2), 312-319.
- Behan, J., McQuinn, K., Roche, M., 2006, "Rural Land Use: Traditional Agriculture or Forestry." *Land Economics*, 82(1), 112-123.
- Black, F., Scholes, M., 1973, "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy*, 81(2), 637-59.
- Boyd, J., Caballero, K., Simpson, D., 1999, "The Law and Economics of Habitat Conservation: Lessons from an Analysis of Easement Acquisitions." Discussion Paper 99-32, Resources for the Future.
- Brown, G., Shogren, J., 1998, "Economics of the Endangered Species Act." *Journal of Economic Perspectives*, 12(3), 3-20.
- Capozza, D., Li, Y., 1994, "The Intensity and Timing of Investment: The Case of Land." *American Economic Review*, 84(4), 889-904.
- Cicchetti, C.J., Freeman, A.M. III, 1971, "Option demand and consumer's surplus: further comment." *Quarterly Journal of Economics*, 85(3), 528-539.
- Clark, H., Reed, W., 1988, "A Stochastic Analysis of Land Development Timing and Property Valuation." *Regional Science and Urban Economics*, 18(3), 357-381.
- Conrad, Jon M., 1997, "On the Option Value of Old Growth Forest." *Ecological Economics*, 22(2), 97-102.
- Conrad, Jon M., 2000, "Wilderness: options to preserve, extract, or develop." *Resource and Energy Economics*, 22, 205-219.
- Cunningham, Christopher R., 2006, "House price uncertainty, timing of development, and vacant land prices: Evidence for real options in Seattle." *Journal of Urban Economics*, 59, 1-31.
- Dixit, A., Pindyck, R., 1994, *Investment Under Uncertainty*. Princeton University Press, Princeton.
- Fisher, Anthony C., 2000, "Investment under uncertainty and option value in environmental economics." *Resource and Energy Economics*, 22, 197-204.

- Forsyth, Margaret, 2000, "On Estimating the Option Value of Preserving a Wilderness Area." *Canadian Journal of Economics*, 33(2), 413.
- Hanemann, W.M., 1989, "Information and the concept of option value." *Journal of Environmental Economics and Management*, 16, 23-37.
- Henry, C., 1974, "Investment decisions under uncertainty: the irreversibility effect." *The American Economic Review*, 64, 1006-1012.
- Innes, R., Polasky, S., Tschirhart, J., 1998, "Takings, Compensation and Endangered Species Protection on Private Lands." *Journal of Economic Perspectives* 12(3). 35-52.
- Insley, M., 2002, "A Real Options Approach to the Valuation of a Forestry Investment." *Journal of Environmental Economics and Management*, 44, 471-492.
- Karlin, Samuel and Howard M. Taylor. 1981. *A Second Course in Stochastic Processes*, 2nd ed. Academic Press, New York.
- Krutilla, John V., 1967, "Conservation Reconsidered." *American Economic Review*, 57(4), 777-786.
- Malchow-Moller, N., Niels, S., Thorsen, B.J., 2004, "Real-options aspects of adjacency constraints." *Forest Policy and Economics* 6, 261-270.
- Margolis, M., Osgood, D.E., List, J., "Does the Endangered Species Act Endanger Species? Evidence from a Natural Experiment." Working paper, 2006.
- McDonald, R., and Siegel, D., 1986, "The Value of Waiting to Invest." *Quarterly Journal of Economics*, 101(4), 707-727.
- Mensink, P., Requate, T., 2004, "The Dixit-Pindyck and the Arrow-Fisher-Hannemann-Henry option values are not equivalent: a note on Fisher (2000)." *Resource and Energy Economics*, 27, 83-88.
- Merton, Robert C., 1973 "Theory of Rational Option Pricing." *The Bell Journal of Economics and Management Science*, 4(1), 141-183.
- Parker, Dominic P., 2005, "Transactions Costs, Nonprofit Incentives, and the Vertical Integration of Land Trusts." Draft, May 19, 2005.
- Parker, D., and Thurman, W., 2004, "Crowding Out Open Space: Federal Land Programs and Their Effects on Land Trust Activity," Working paper, September, 2004.

Plantinga, Andrew, 2006, "The Economics of Conservation Easements." Draft Paper prepared for the Lincoln Institute of Land Policy 'Land Policies for Urban Development' Conference.

Pindyck, Robert S., 1991, "Irreversibility, Uncertainty, and Investment." *Journal of Economic Literature*, Vol. 29, No. 3 (Sep., 1991), pp. 1110-1148.

Pindyck, Robert S., 2000, "Irreversibilities and the Timing of Environmental Policy." *Resource and Energy Economics*, 22(3), 233-259.

Pindyck, Robert S., 2002, "Optimal Timing Problems in Environmental Economics." *Journal of Economic Dynamics and Control*, 26(9-10), 1677-1697.

Quigg, Laura, 1993, "Empirical Testing of Real Option-Pricing Models." *The Journal of Finance*, 48(2), 621-640.

Riddiough, Timothy J., 1997, "The Economic Consequences of Regulatory Taking Risk on Land Value and Development Activity." *Journal of Urban Economics* 41, 56-77.

Tegene, A., Wiebe, K., Kuhn, B., 1999, "Irreversible Investment Under Uncertainty: Conservation Easements and the Option to Develop Agricultural Land." *Journal of Agricultural Economics*, 50(2), 203-219.

Titman, Sheridan, 1985, "Urban Land Prices Under Uncertainty." *American Economic Review*, 75(3), 505-514.

Wiemers, E., Behan, J., 2004, "Farm Forestry Investment in Ireland Under Uncertainty." *The Economic and Social Review*, 35(3), 305-320.

# Appendix

## One-time Offer Model

For the one time offer problem, comparative statics can be determined analytically:

$$\partial\beta_2/\partial r = 1/((\beta_2 - 1/2)\sigma^2 + \mu) > 0$$

$$\partial\beta_2/\partial\mu = -\beta_2/((\beta_2 - 1/2)\sigma^2 + \mu) < 0$$

$$\partial\beta_2/\partial\sigma = -\beta_2(\beta_2 - 1)\sigma/((\beta_2 - 1/2)\sigma^2 + \mu) < 0$$

Also  $d\beta_2/(\beta_2 - 1)/d\beta_2 = -1/(\beta_2 - 1)^2 < 0$ . These expressions can be used to show that for development cutoff price  $\partial P^*/\partial r < 0$ ,  $\partial P^*/\partial\mu > 0$  and  $\partial P^*/\partial\sigma > 0$ . Also  $\partial P^*/\partial A = P^*/A > 0$ . When the model is extended to include the possibility of regulatory takings we obtain the result that

$$\frac{\partial P^*}{\partial\lambda} = -\frac{1}{(\beta_2 - 1/2)\sigma^2 + \mu} \frac{A}{r} < 0$$

The minimal easement value needed to induce acceptance is  $E^*(P) = \alpha_2 P^{\beta_2} = \frac{1}{\beta_2} P^{\beta_2} P^{*(1-\beta_2)}$  when  $P < P^*$ . For a given level of  $P$  the derivative of the option value with respect to  $\beta_2$  is

$$\frac{dE^*}{d\beta_2} = -\alpha_2 P^{\beta_2} \ln(P/P^*) < 0$$

so  $\partial E^*/\partial\mu > 0$  and  $\partial E^*/\partial\sigma > 0$ . The sign of  $\partial E^*/\partial r$ , however, is ambiguous. Also

$$\frac{\partial E^*}{\partial A} = -\frac{1}{r} \left(\frac{P}{P^*}\right)^{\beta_2} < 0$$

## Standing Offer Model

For the two sided optimal stopping problem associated with standing offers the solution cannot be expressed in closed form but must be solved numerically. The value matching

conditions can be written as

$$\begin{bmatrix} P_E^{\beta_1} & P_E^{\beta_2} \\ P_D^{\beta_1} & P_D^{\beta_2} \end{bmatrix} \eta = \begin{bmatrix} E \\ P_D - A/r \end{bmatrix}$$

and the smooth-pasting conditions as

$$\begin{bmatrix} \beta_1 P_E^{\beta_1-1} & \beta_2 P_E^{\beta_2-1} \\ \beta_1 P_D^{\beta_1-1} & \beta_2 P_D^{\beta_2-1} \end{bmatrix} \eta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Defining  $z = P_E/P_D$ , and solving for  $\eta$ , yields the conditions

$$\begin{aligned} & \begin{bmatrix} \beta_1 & \beta_2 z^{\beta_2} \\ \beta_1 z^{-\beta_1} & \beta_2 \end{bmatrix} \begin{bmatrix} 1 & z^{\beta_2} \\ z^{-\beta_1} & 1 \end{bmatrix}^{-1} \begin{bmatrix} E \\ P_D - A/r \end{bmatrix} \\ &= \frac{1}{1-z^{\beta_2-\beta_1}} \begin{bmatrix} \beta_1 - \beta_2 z^{\beta_2-\beta_1} & (\beta_2 - \beta_1) z^{\beta_2} \\ (\beta_1 - \beta_2) z^{-\beta_1} & \beta_2 - \beta_1 z^{\beta_2-\beta_1} \end{bmatrix} \begin{bmatrix} E \\ P_D - A/r \end{bmatrix} = \begin{bmatrix} 0 \\ P_D \end{bmatrix} \end{aligned}$$

These equations are linear in  $P_D$ ; using the first expression yields

$$P_D = A/r + \frac{(\beta_2 z^{\beta_2-\beta_1} - \beta_1) E}{(\beta_2 - \beta_1) z^{\beta_2}}$$

Inserting this into the second expression yields a root condition for  $z$ :

$$\begin{aligned} 0 &= \left( \beta_2 - 1 + (1 - \beta_1) z^{\beta_2-\beta_1} \right) \left( (\beta_2 z^{\beta_2-\beta_1} - \beta_1) E + (\beta_2 - \beta_1) z^{\beta_2} \frac{A}{r} \right) \\ &\quad - \left( (\beta_2 - \beta_1) z^{-\beta_1} E + (\beta_2 - \beta_1) z^{\beta_2-\beta_1} \frac{A}{r} \right) (\beta_2 - \beta_1) z^{\beta_2} \end{aligned}$$

Given that  $\beta_1 < 0 < 1 < \beta_2$  for economically relevant values of  $\mu$ ,  $\sigma$  and  $r$ , this root condition is stable over most parameter values. In particular,  $z$  is bounded on  $[0, 1)$  and is always raised to positive powers.<sup>10</sup>

---

<sup>10</sup>There is a trivial solution to this system of equations, namely  $P_D = E + A/r$  and  $z = 1$ , but this is not the solution we seek (it implies division by 0 in the above derivation).