Linear Inverse Problems

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OUR PLAN

Review Pertinet Linear Algebra Topics

Forward Problem for Linear Systems

Inverse Problem for Linear Systems

Discuss Well-posedness and Overdetermined Systems

Formulate a least squares solution for an overdetermined system

Introduction

Linear Algebra Review Represent m linear equations with n variables:

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{bmatrix}$$

A=m x n matrix, x=n x 1 vector b=m x 1 vector

If A=m x n and B=n x p then AB=m x p
(number of columns of A = Number of rows of B)

Matrix Properties

Matrix Properties:
A= Square matrix has n rows and n columns
if A is square then A^{-1} exists iff the determinant $\neq 0$

What is a determinant??

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Determinant=$ad - bc$ How do we find the inverse once we determine that the determinant is $\neq 0$?

$$\frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Solve the following system of linear equations :

Refer to your Worksheet problem 1

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Solving this by hand is simple...
Let $b_1 = 1$ and $b_2 = 3$

Then our system of linear equations is:
$2x_1 + x_2 = 1$
$x_1 + 3x_2 = 3$

other properties

if A^{-1} exists then $A^{-1}A = I$

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

EXAMPLE: Square Matrix

$$\begin{bmatrix} x_1 - x_2 + 3x_3 + x4 = 2 \\ 3x_1 - 3x_2 + x_3 \qquad = -1 \\ x_1 + x_2 \qquad - 2x_4 = 3 \\ x_1 + x_2 + x_3 - x_4 = 1 \end{bmatrix}$$

Solving this by hand is tedious and very time consuming,

especially as m and n grow larger

HOW DO WE SOLVE?

OUR EXAMPLE

$$A = \begin{bmatrix} 1 & -1 & 3 & 1 \\ 3 & -3 & 1 & 0 \\ 1 & 1 & 0 & -2 \\ 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}$$

SOLVE USING MATLAB and the INVERSE of A

Forward Problem

The forward problem is fairly straightforward

Ax=b

If we have A an $n * n$ matrix and x an $n * 1$ vector then it is clear how we will solve for b

The forward problem consists of finding b for a given A and x

Example

What if

$$A = \begin{bmatrix} 47 & 28 \\ 89 & 53 \end{bmatrix} \text{ and } x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Solving for b is fairly straightforward.

$b_1 = 47 - 28$
$b_2 = 89 - 53$

Inverse Problem

For the Vibrating Beam, we are given data

(done in the lab tomorrow)

and we must determine m, c and k.

In the case of the linear system $Ax = b$ we are provided

with A and b and must determine x

Example

$$Ax = b \rightarrow A^{-1}Ax = A^{-1}b \rightarrow x = A^{-1}b$$

$$\begin{bmatrix} 0 & 1 & -1 \\ -2 & 4 & -1 \\ -2 & 5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -2 & 4 & -1 \\ -2 & 5 & -4 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

We will discuss in the following section, using the BACKSLASH operator

Alternative Method to solve

Matrix Calculations

Ax=b
MULTIPLY BOTH SIDES BY A^{-1}

$A^{-1}Ax = A^{-1}b$

$Ix = A^{-1}b$

NOTE: $Ix = x$

NOW IN MATLAB

These calculations can be performed using the BACKSLASH OPERATOR in MATLAB

MATLAB- x= A $\backslash b$

!!!NOTE: This is not the same as A divided by b!!!

PROBLEM

What if A is not a square matrix???

A is no longer directly invertible

(no longer a set of unique solutions to the problem)

BUT we can minimize the equation using method of least squares to be shown later this afternoon

PROBLEM:

A is no longer a square matrix
We wish to solve $Ax = b$

- Focus on an overdetermined System: (i.e. A is $m * n$ where $m > n$)

- Usually no exact solution exists when A is overdetermined

- Definition. A linear system is called inconsistent or overdetermined if it does not have a solution. In other words, the set of solutions is empty. Otherwise the linear system is called consistent.

- In our experiment this week, the number of data points will exceed the number of parameters to solve $m > n$

Well Posed

What does it mean to be Well-Posed?
$Ax = b$ is well posed when
Existence - for every b there exists a x such that $Ax = b$
Uniqueness - if $Ax_1 = Ax_2 \rightarrow x_1 = x_2$
Stability - A^{-1} is continuous

The solution technique $x = A^{-1}b$ produces the correct answer when $Ax = b$ is *well-posed*
$Ax = b$ is ill-posed if it is not well-posed

Example

x is the solution to $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

See Matlab worksheet:

Ill-Posedness

x is the solution to $Hx = \vec{1}$ where H is the HILBERT MATRIX

TRY IT OUT IN MATLAB (WORKSHEET)

What is an Ill conditioned System

A system is ill-conditioned if some small perturbation in the system causes a relatively large change in the exact solution

EXAMPLE

Try entering this into your matrix and solving:

$$\begin{bmatrix} .835 & .667 \\ .333 & .266 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} .168 \\ .067 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} .835 & .667 \\ .333 & .266 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} .168 \\ .066 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

YOU WILL NOTICE MAJOR CHANGES HERE!

Overdetermined System

- Example $\|\vec{x}\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{x_1^2 + x_2^2 + + x_n^2}$
- MINIMIZE $\|Ax - b\|_2^2 = (Ax - b)^T (Ax - b)$

Obtaining the Normal Equations

- We want to minimize

$\phi(x) = (Ax - b)^T(Ax - b)$:

$\nabla\phi(x) = A^T(Ax - b) + \left((Ax - b)^T A\right)^T$

$= A^T(Ax - b) + A^T(Ax - b)$

$= A^T Ax - A^T b + A^T Ax - A^T b$

$= 2(A^T Ax - A^T b)$

- $\phi(x)$ is minimized when x solves $A^T Ax = A^T b$
- $x = (A^T A)^{-1} A^T b$ provides the *least squares solution*

Which is a topic that will be presented later on today!