Solving the Harmonic Oscillator

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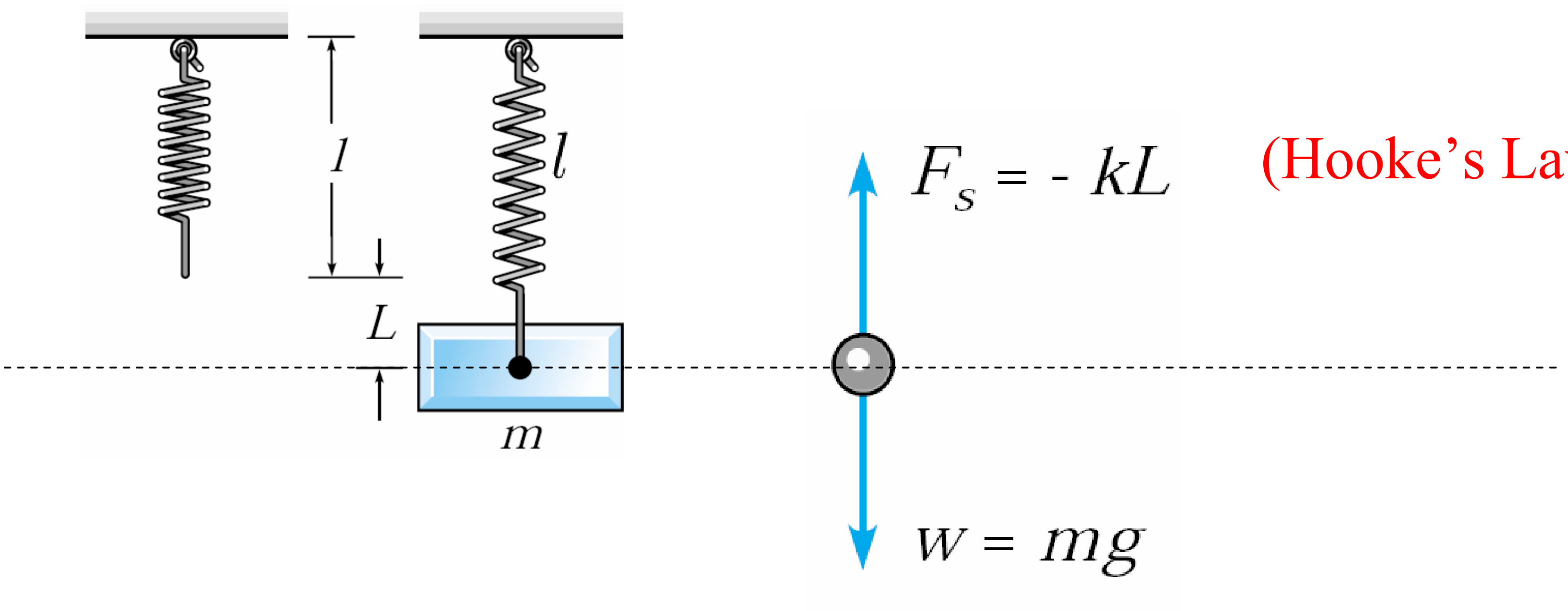
Contents

1. Simple illustrative example: Spring-mass system
2. Free Vibrations: Undamped
3. Free Vibrations: Damped
4. Forced Vibrations: Beats and Resonance
5. Will this work for the beam?
6. Writing as a First Order System
7. Summary & References

1. Spring-mass system

What is a spring-mass system and why it is important?



(Hooke's Law)

w = gravitational force

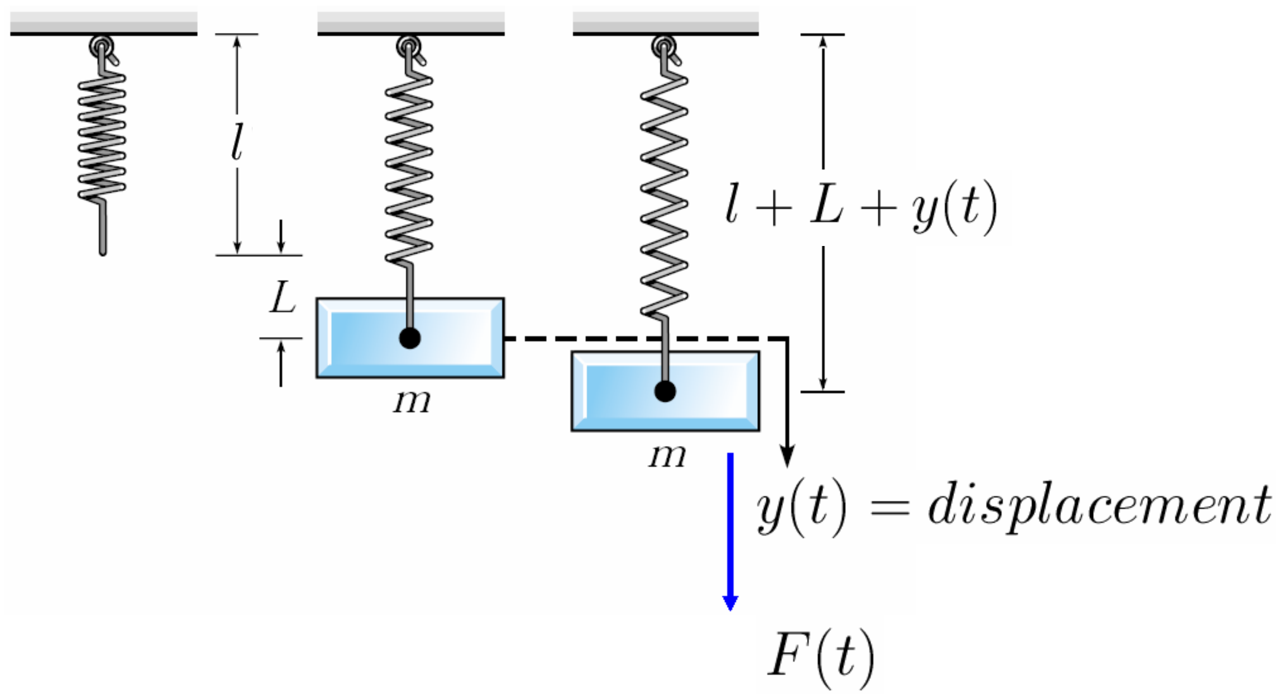
F_s = spring force

g = gravitational acceleration

k = spring constant

1. Spring-mass system

Dynamic problem: What is motion of the mass when acted by an external force or is initially displaced?



1. Spring-mass system

Forces acting on the mass

- the applied force $F(t)$
- the weight $w = mg$
- the spring force $F_s(t) = -k[L + y(t)]$
- the damping force $F_d(t) = -cy'(t)$

Net force acting on the mass

$$\begin{aligned} F_{net}(t) &= F(t) + w + F_s(t) + F_d(t) \\ &= F(t) + \cancel{mg} - \cancel{kL} - ky(t) - cy'(t) \\ &= F(t) - ky(t) - cy'(t) \end{aligned}$$

1. Spring-mass system

Newton's Second Law of Motion

the acceleration of an object due to an applied force is in the direction of the force and given by:

$$F_{net} = ma = my''(t)$$

For our spring-mass system

$$my''(t) = \underbrace{F(t) - cy'(t) - ky(t)}_{F_{net}}$$

$$\boxed{my''(t) + cy'(t) + ky(t) = F(t)}$$

2. Undamped Free Vibrations

$$my''(t) + \cancel{cy'(t)} + ky(t) = \cancel{F(t)}$$

no damping — no external force

$$my''(t) + ky(t) = 0$$

$$y''(t) + \omega_0^2 y(t) = 0 \qquad \omega_0^2 = \frac{k}{m}$$

$$\cos\omega_0 t, \quad \sin\omega_0 t \qquad \text{(particular solutions)}$$

$$y(t) = A\cos\omega_0 t + B\sin\omega_0 t \qquad \text{(general solution)}$$

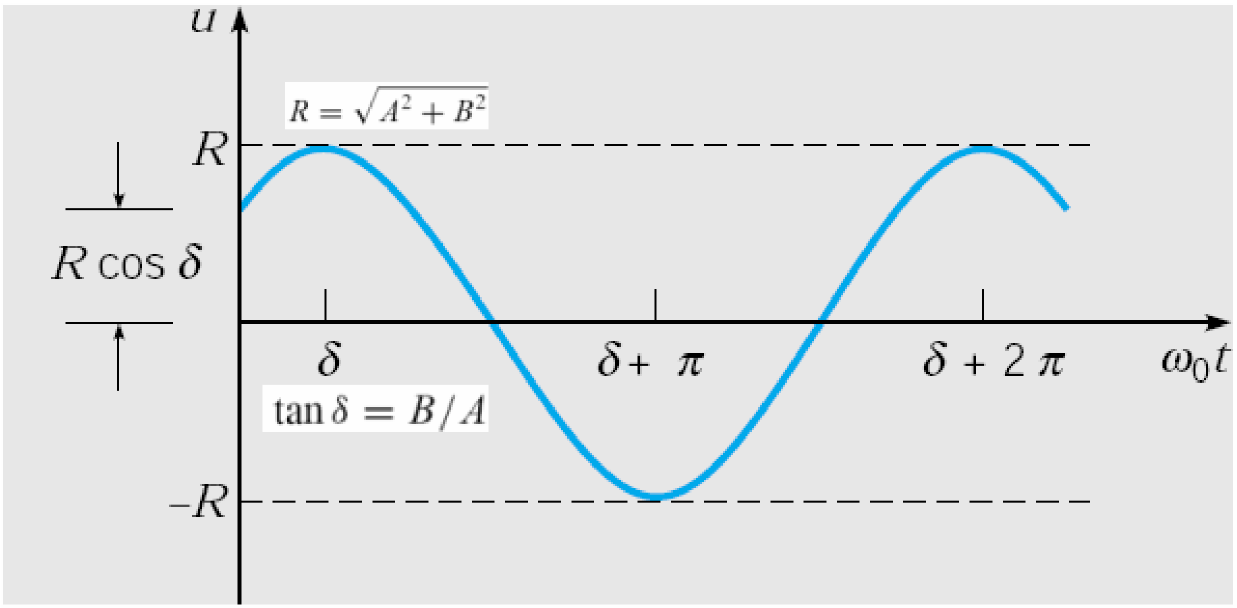
A and *B* are arbitary constants determined from initial conditions $y(0) = y_0$ and $y'(0) = y_0'$

2. Undamped Free Vibrations

$$y(t) = A\cos\omega_0 t + B\sin\omega_0 t$$

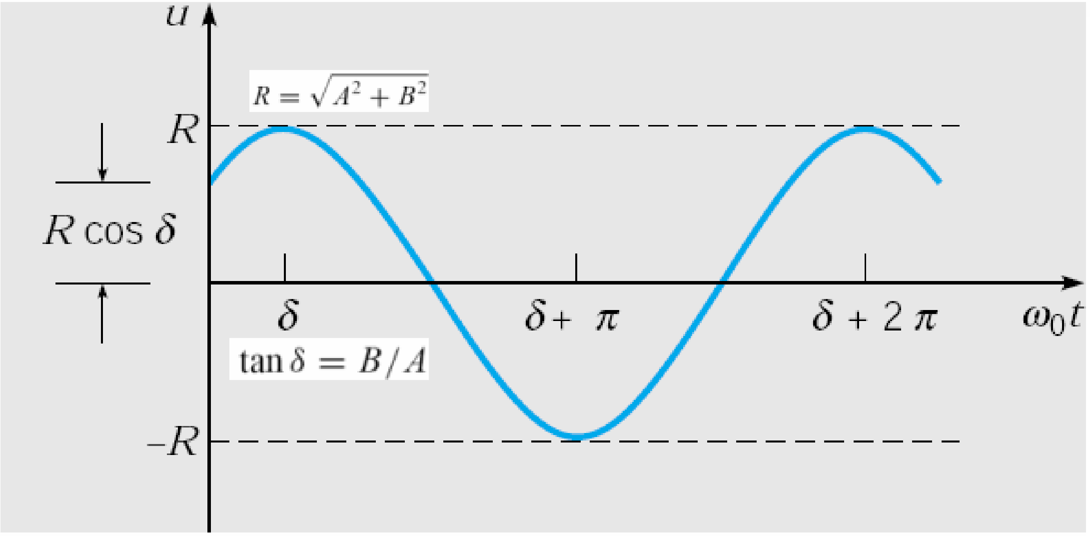
$$y(t) = R\cos(\omega_0 t - \delta)$$

$$R = \sqrt{A^2 + B^2}, \qquad \tan\delta = B/A.$$



Periodic, simple harmonic motion of the mass

2. Undamped Free Vibrations



$$T = \frac{2\pi}{\omega_0} = 2\pi \left(\frac{m}{k}\right)^{1/2}$$ **Period** of motion

$$\omega_0 = \sqrt{\frac{k}{m}}$$ **Natural frequency** of the vibration

$$R = \sqrt{A^2 + B^2}$$ **Amplitude** (constant in time)

$$\delta$$ **Phase** or phase angle

3. Damped Free Vibrations

$$my''(t) + cy'(t) + ky(t) = \cancel{F(t)}$$

no external force

$$my''(t) + cy'(t) + ky(t) = 0$$

Assume an exponential solution $y(t) = e^{rt}$

Then

$$y'(t) = re^{rt}, \quad y''(t) = r^2 e^{rt}$$

and substituting in equation above, we have

$$\boxed{mr^2 + cr + k = 0}$$

(characteristic equation)

3. Damped Free Vibrations

$$mr^2 + cr + k = 0$$

Solutions to characteristic equation:

$$r_1, r_2 = \frac{-c \pm \sqrt{c^2 - 4km}}{2m} = \frac{c}{2m}\left(-1 \pm \sqrt{1 - \frac{4km}{c^2}}\right)$$

$c^2 - 4km > 0, \quad y = Ae^{r_1 t} + Be^{r_2 t}$ overdamped

$c^2 - 4km = 0, \quad y = (A + Bt)e^{-ct/2m}$ critically damped

$c^2 - 4km < 0, \quad y = e^{-ct/2m}(A\cos\mu t + B\sin\mu t)$ underdamped

m, c, k are positive ⟶ The solution ***y*** decays as t goes to infinity *regardless* the values of ***A*** and ***B***

Damping gradually dissipates energy!

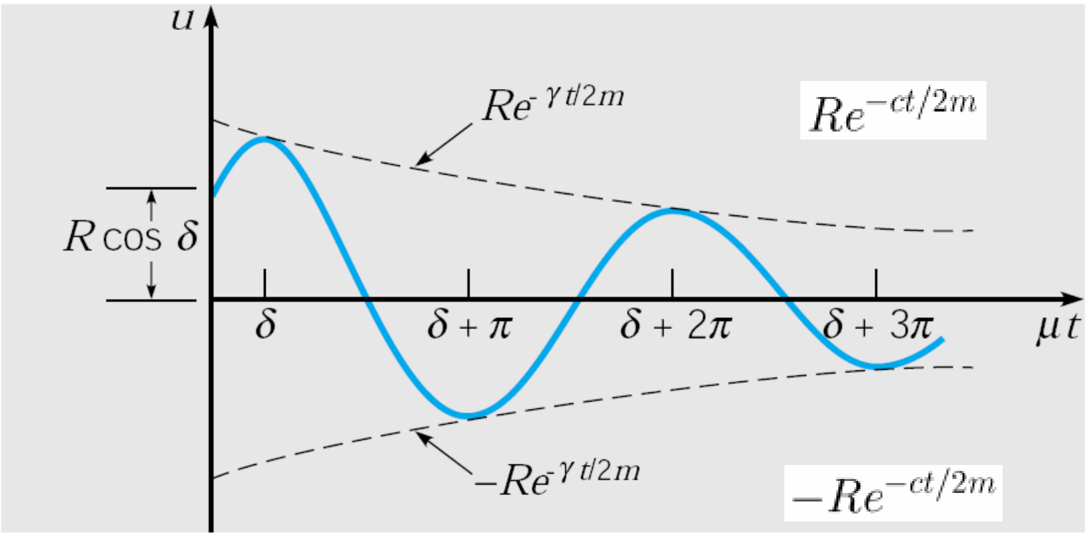
3. Damped Free Vibrations

The most interesting case is ***underdamping***, i.e:

$$y = e^{-ct/2m}(A\cos\mu t + B\sin\mu t)$$

$$A = R\cos\delta \text{ and } B = R\sin\delta$$

$$\boxed{y = Re^{-ct/2m}\cos(\mu t - \delta)}$$



$$\mu = \frac{(4km - c^2)^{1/2}}{2m}$$

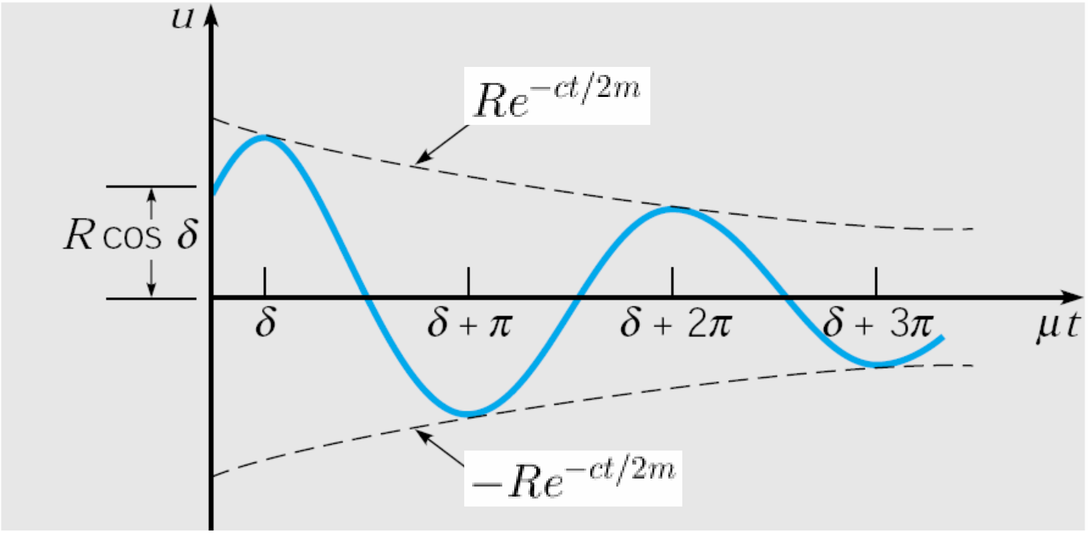
3. Damped Free Vibrations: Small Damping

$$y = Re^{-ct/2m}\cos(\mu t - \delta)$$

$$\mu = \frac{(4km - c^2)^{1/2}}{2m}$$

$\mu = quasi \quad frequency$

$$\frac{\mu}{\omega_0} = 1 - \frac{c^2}{8km}$$



$$T_d = \frac{2\pi}{\mu}$$

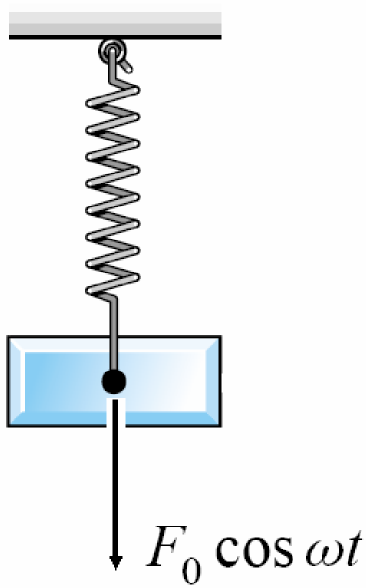
$T_d = quasiperiod$

$$\frac{T_d}{T} = \frac{\omega_0}{\mu} = \left(1 - \frac{c^2}{4km}\right)^{-1/2} \cong \left(1 + \frac{c^2}{8km}\right)$$

As $\frac{c^2}{4km}$ increases $\Longrightarrow$ the quasi frequency μ decreases

$\Longrightarrow$ the quasi period T_d increases

4. Forced Vibrations



$$my''(t) + cy'(t) + ky(t) = \underbrace{F(t)}_{F_0 \cos \omega t}$$

<u>Periodic external force:</u>

$$my''(t) + \cancel{cy'(t)} + ky(t) = F_0 \cos \omega t$$

no damping

$$my''(t) + ky(t) = F_0 \cos \omega t$$

<u>Case 1</u> $\omega_0 = \sqrt{k/m} \neq \omega$

$$y = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$$

4. Forced Vibrations: Beats

$$y = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$$

$y(0) = 0$ and $y'(0) = 0$ ⟶ $c_1 = -\frac{F_0}{m(\omega_0^2 - \omega^2)}, \quad c_2 = 0,$

$$y = \frac{F_0}{m(\omega_0^2 - \omega^2)} (\cos \omega t - \cos \omega_0 t)$$

$$y = \left[\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{(\omega_0 - \omega)t}{2} \right] \sin \frac{(\omega_0 + \omega)t}{2}$$

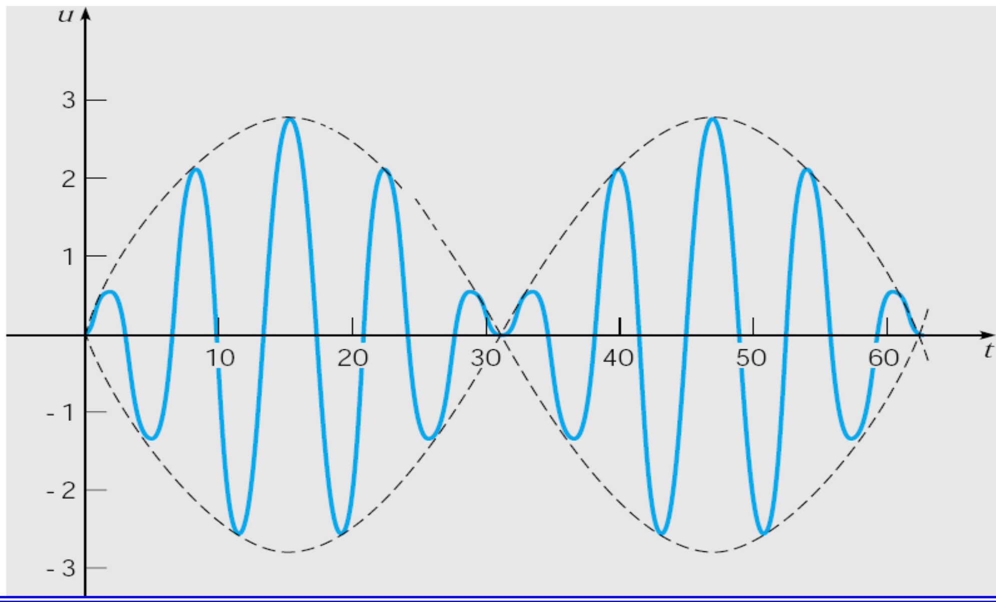
4. Forced Vibrations: Beats

ω_0 is close ω

$$y = \left[\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\frac{(\omega_0 - \omega)t}{2}\right] \sin\frac{(\omega_0 + \omega)t}{2}$$

Slowly oscillating amplitude

Rapidly oscillating

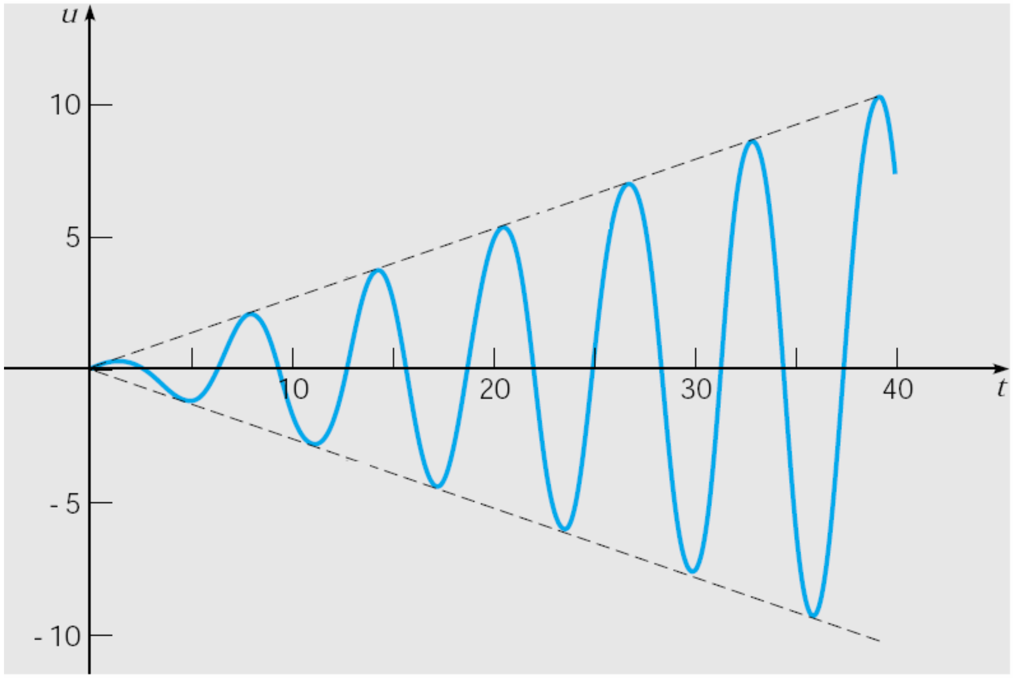


4. Forced Vibrations: Resonance

$$my''(t) + ky(t) = F_0 \cos \omega t$$

Case 2 $\quad \omega_0 = \sqrt{k/m} = \omega$

$$y = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{2m\omega_0} t \sin \omega_0 t$$



unbounded as $t \to \infty$

5. Will this work for the beam?

- The beam seems to fit the harmonic conditions
  - Force is zero when displacement is zero
  - Restoring force increases with displacement
  - Vibration appears periodic
- The key assumptions are
  - Restoring force is linear in displacement
  - Friction is linear in velocity

6. Writing as a First Order System

- Matlab does not work with second order equations

- However, we can always rewrite a second order ODE as a system of first order equations

- We can then have Matlab find a numerical solution to this system

6. Writing as a First Order System

Given the second order ODE

$$my''(t) + cy'(t) + ky(t) = 0$$

with the initial conditions

$$y(0) = y_0 \quad \text{and} \quad y'(0) = y'_0$$

We can let $z_1(t) = y(t)$ and $z_2(t) = y'(t)$. Clearly

$$z'_1(t) = z_2(t)$$
$$z'_2(t) = -kz_1(t) - cz_2(t)$$

6. Writing as a First Order System

Now we can rewrite the equation in matrix-vector form

$$\begin{bmatrix} z_1'(t) \\ z_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$

$$\mathbf{z}'(\mathbf{t}) = \mathbf{A}\mathbf{z}(\mathbf{t})$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix} \qquad \mathbf{z_0} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$$

This is a format Matlab can handle.

6. Constants are not independent

- Notice that in all our solutions we never have c, m, or k alone. We always have c/m or k/m.

- □The solution for y(t) given (m,c,k) is the same as y(t) given (αm, αc, αk).

- important for the inverse problem

7. Summary

- We can use Matlab to generate solutions to the harmonic oscillator
- At first glance, it seems reasonable to model a vibrating beam
- We don't know the values of m, c, or k
- Need to solve the inverse problem

8. References

W. Boyce and R.C. DiPrima: Elementary Differential Equations and Boundary Value Problems