

Random Matrix Theory

- *What is a Random Matrix?* A matrix whose elements x_{ij} are random variables

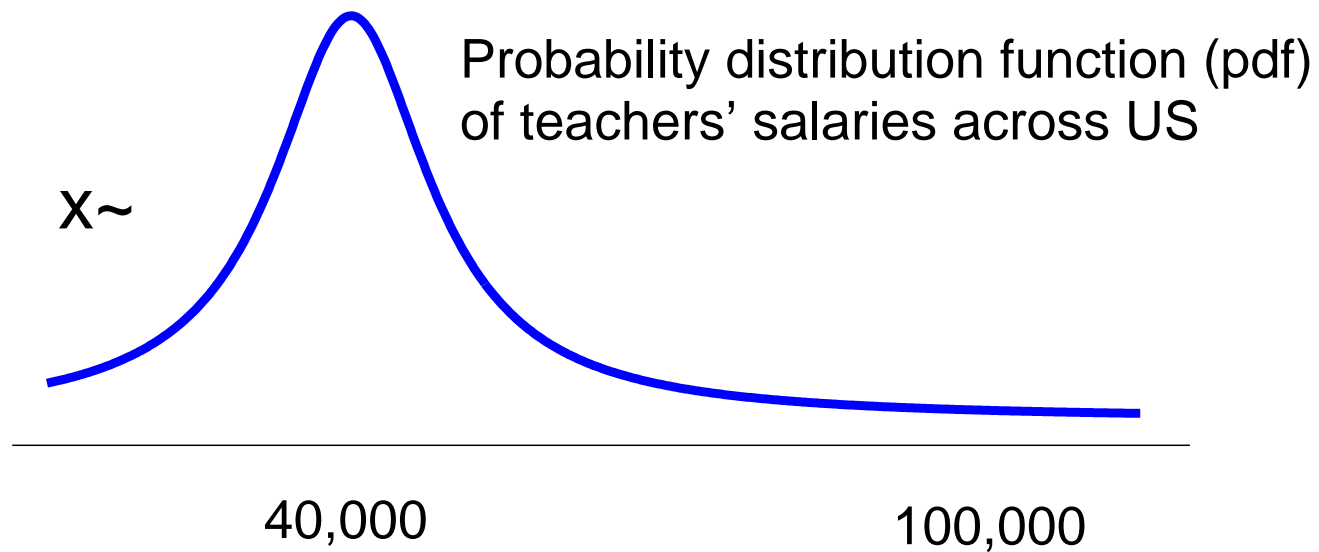
$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{p1} & x_{p1} & \cdots & x_{pp} \end{pmatrix}$$

- *What is a random variable?* A quantity whose value is random and to which a probability distribution is assigned
- RV Example (discrete): Number of siblings

x	0	1	2	3	...
Pr	0.28	0.3	0.24	0.08	...

Random Matrix Theory

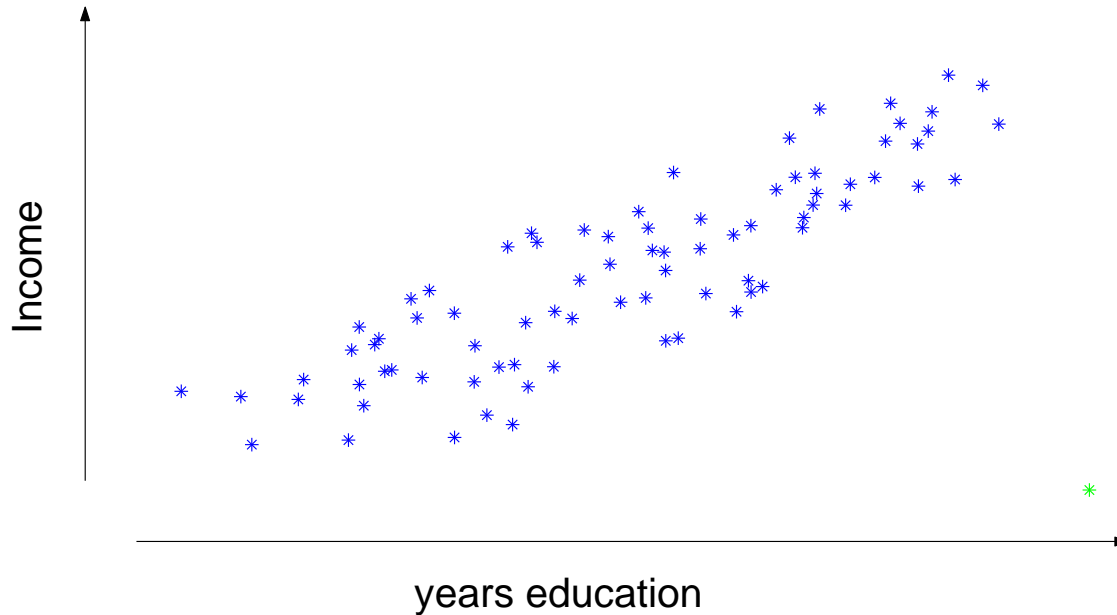
- RV Example (continuous): Annual teaching income across US



Random Matrix Theory

Natural tools which helps us explore relationships between RVs

- RM Example: education (x_1) and income (x_2)



This relationship can be express as a random matrix

$$\begin{pmatrix} \text{var}[x_1] & \text{cov}[x_1, x_2] \\ \text{cov}[x_2, x_1] & \text{var}[x_2] \end{pmatrix}$$

Random Matrix Theory

Other applications:

- Wireless communications
- Nuclear physics
- Finance
- Genetics
- . . .

Rare events in nonlinear lightwave systems

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SAMSI/CRSC undergraduate workshop - May 21, 2007

Communications

Goal

- send lots of information very quickly over very long distances
- make almost no mistakes

Optical Communications

- use light to represent information
- transmit light (information) through fiber

Our goal

- model this system mathematically
- calculate the probability errors

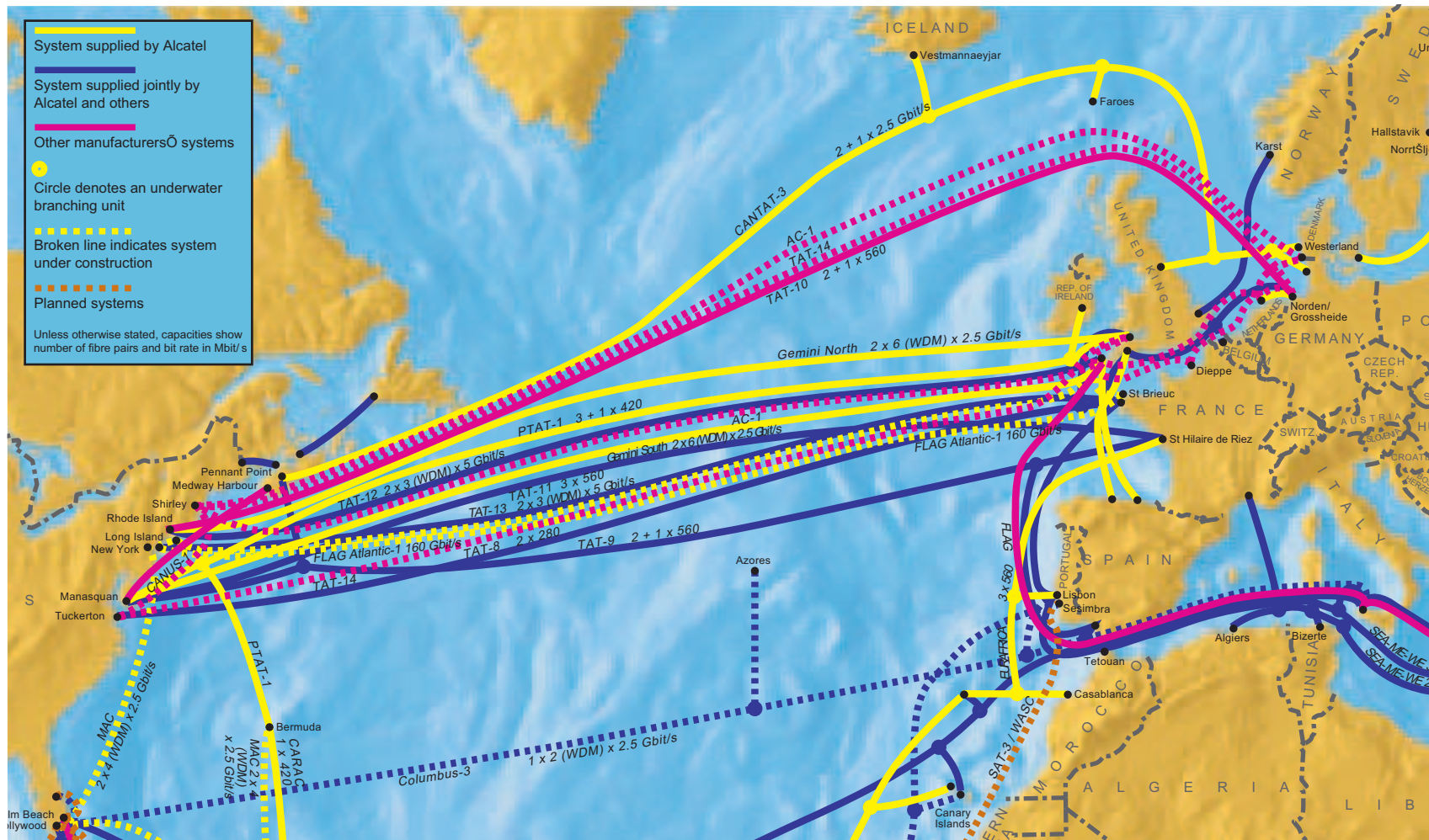
Very brief history of optical fiber communications

Fiber Optic Communication: The transmission of information via a signal comprised of light that travels through glass fiber which acts as a wave guide.

- made possible by **laser** (1960) and **low-loss**, single-mode fiber (1970)
- nonlinear Schrödinger equation exactly solvable by the inverse scattering transform: Zakharov and Shabat, (1971)
- **NLS** proposed for fibers by Hasegawa and Tappert, (1973)
- experimental verification by Mollenauer, (1983)
- made practical by advent of **in-line fiber amplifiers** (1988)

Optical fiber communication systems: large scale

Optical Fibre Submarine Systems North Atlantic

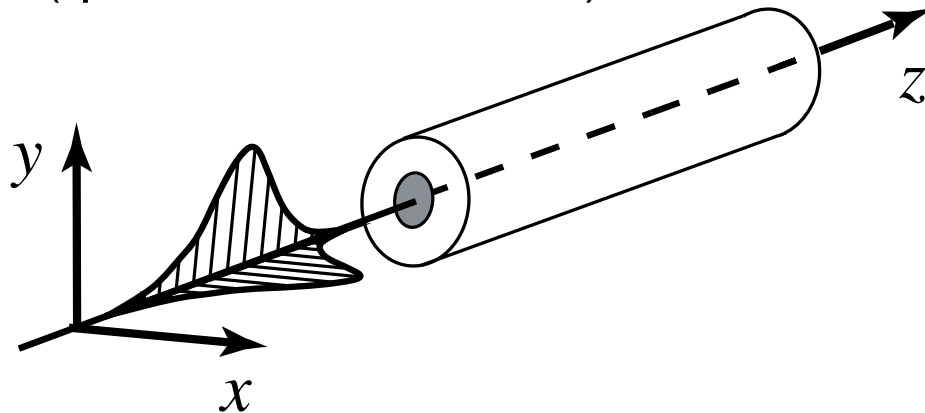


[2002]

Propagation of optical pulses in fibers: From Maxwell's equations to the nonlinear Schrödinger equation

fiber properties

- single mode fiber (only one transverse pulse shape)
- intensity-dependent index of refraction (**nonlinearity**)
- slowly varying envelope approx (quasi monochromatic)
- small, anomalous dispersion



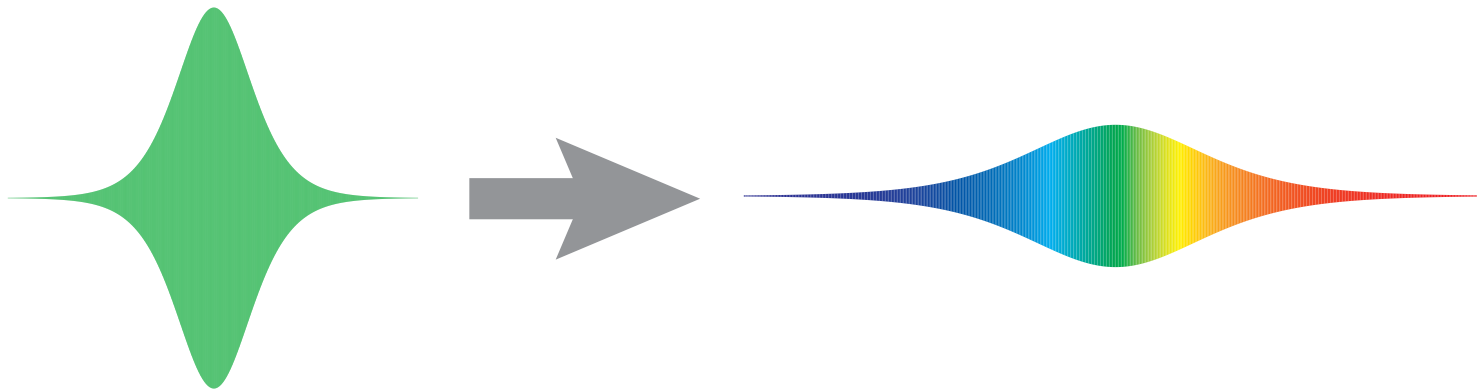
$$\text{NLS: } iu_z + \frac{1}{2} u_{tt} + |u|^2 u = 0$$

Define signal in time (one pulse per “bit slot”), propagate in space

The first impairment is dispersion

$$\frac{\partial u}{\partial z} = \frac{i}{2} \frac{\partial^2 u}{\partial t^2}$$

2nd derivative makes pulse widen (disperse) with distance since different frequencies have different group velocities

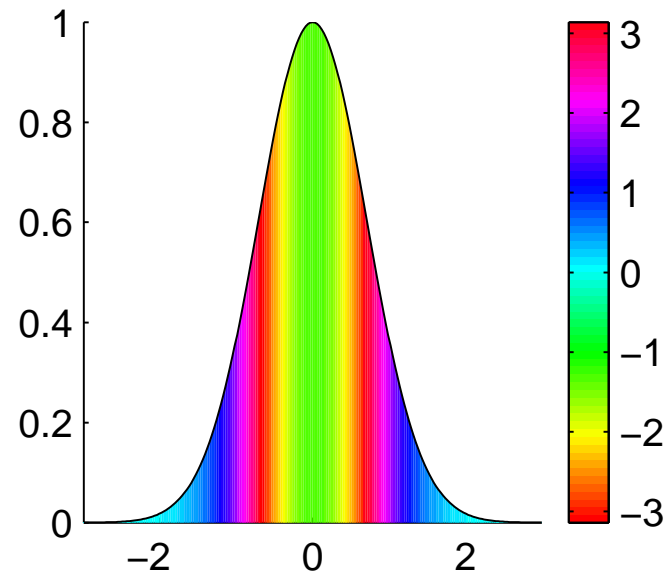
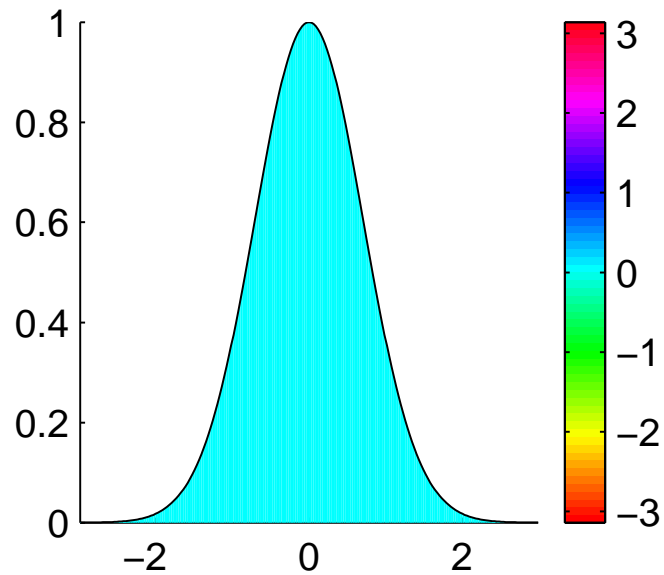


time dependent phase variations \Rightarrow “chirp” or instantaneous differences in frequency across the pulse

The second impairment is nonlinearity

$$\frac{\partial u}{\partial z} = i|u|^2 u$$

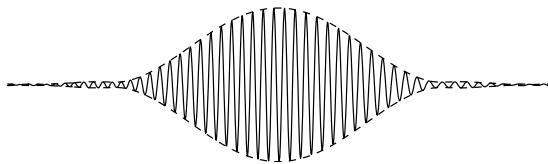
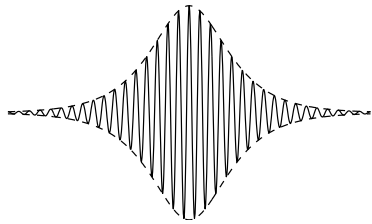
- Intensity-dependent phase rotation (as below)



Nonlinear dispersive waves: an example

dispersion

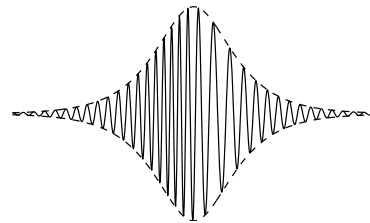
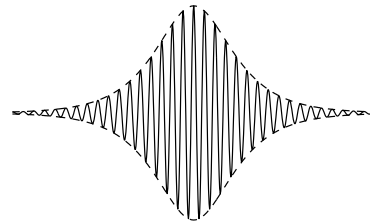
$$iu_z + u_{tt} = 0$$



pulse spreading

nonlinearity

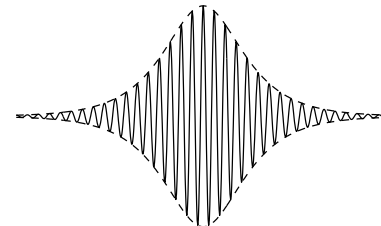
$$iu_z + |u|^2 u = 0$$



self-steepening, shocks

dispersion+nonlinearity

$$iu_z + u_{tt} + |u|^2 u = 0$$



?

Some invariances of the NLS equation

NLS: $iu_z + \frac{1}{2}u_{tt} + |u|^2u = 0$ is invariant to

- scale (amplitude) changes
- phase rotations
- “timing” (position) shifts
- Galilean (velocity) shifts

Example: phase Since NLS is **invariant to phase rotations** if u_s is a solution to NLS, then $u_s e^{i\theta}$ is also a solution to NLS. **Let's check**

$$i \frac{\partial(u_s e^{i\theta})}{\partial z} + \frac{1}{2} \frac{\partial^2(u_s e^{i\theta})}{\partial t^2} + u_s e^{i\theta} u_s^* e^{-i\theta} u_s e^{i\theta} = \left(i \frac{\partial u_s}{\partial z} + \frac{1}{2} \frac{\partial^2 u_s}{\partial t^2} + |u_s|^2 u_s \right) e^{i\theta} = 0$$

Solitons and solitary waves

- “Solitons” = solitary waves that interact “elastically”.

$$u_s(t, z) = A \operatorname{sech}[S(t, z)] \exp[i\varphi(t, z)]$$

$$S(t, z) = A(t - T - \Omega z),$$

$$\varphi(t, z) = \Omega t - \frac{1}{2}(\Omega^2 - A^2)z + \phi.$$

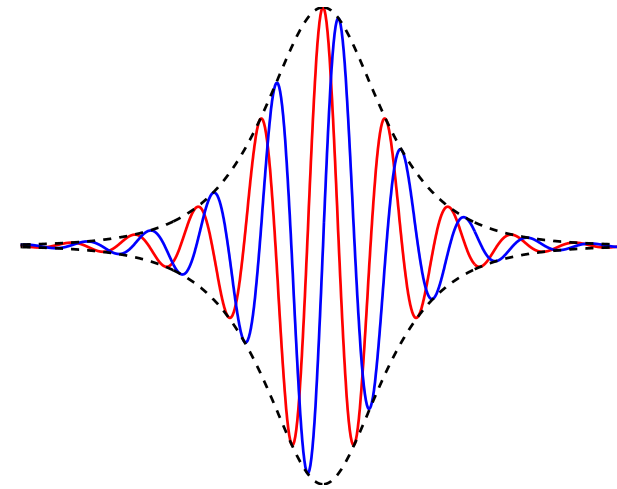
Soliton parameters:

$A \leftrightarrow$ amplitude

$T \leftrightarrow$ timing

$\Omega \leftrightarrow$ frequency shift

$\phi \leftrightarrow$ phase



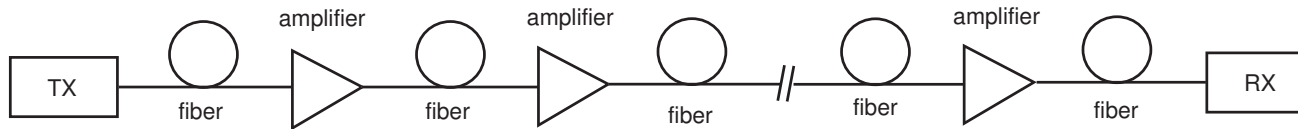
red: $\operatorname{Re}\{u_s\}$

blue: $\operatorname{Im}\{u_s\}$

black: $|u_s|$

- Any combination of parameters results in a valid solution
- How do perturbations changes this solution?

Noise in nonlinear lightwave systems



- Signal amplification by stimulated emission is always accompanied by spontaneous emission of incoherent photons \Rightarrow **noise**

$$iu_z + \frac{1}{2} u_{tt} + |u|^2 u = \sum_{k=1}^{N_{amp}} n_k(t) \delta(z - kz_a)$$

That is, $u(kz_a^+, t) = u(kz_a^-, t) + n_k(t)$, where $n_k(t)$ is i.i.d. Gaussian

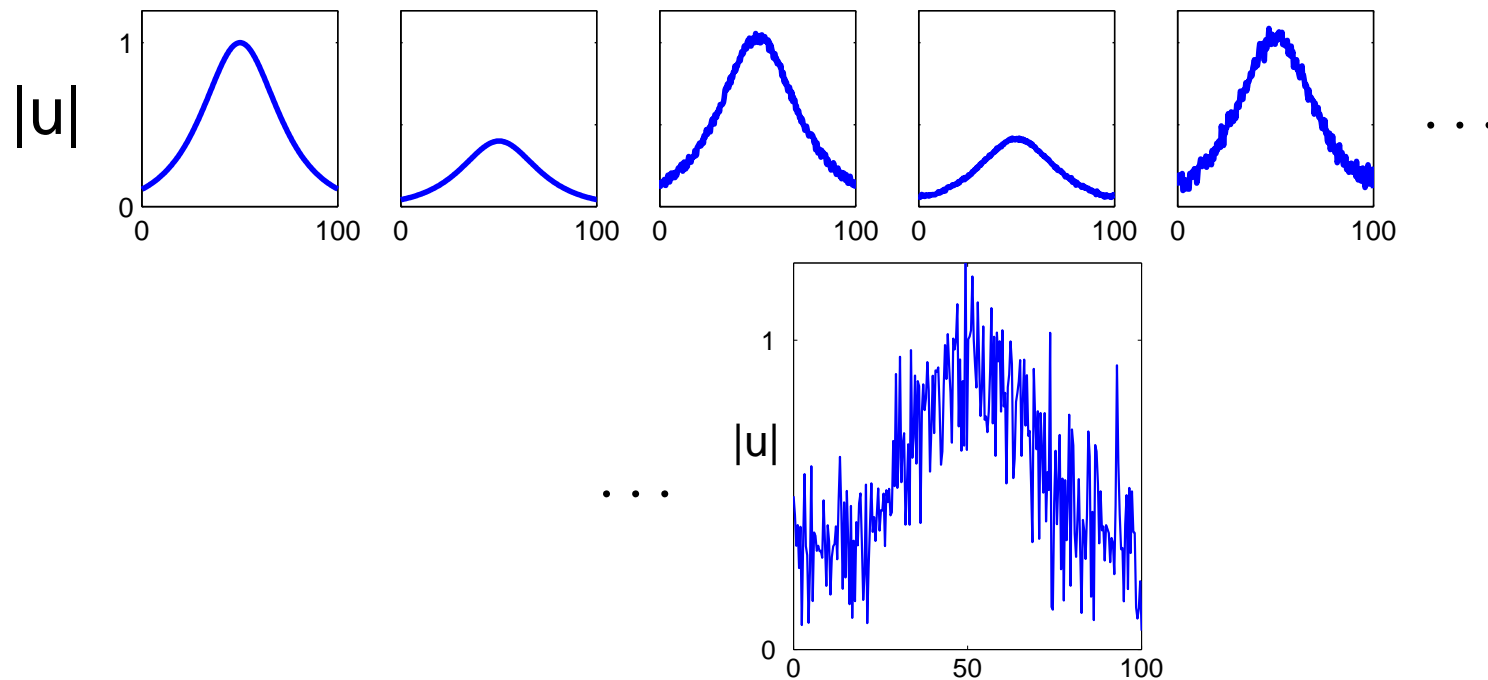
$$\langle n_k(t) \rangle = 0,$$

$$\langle n_k(t) n_j^*(t') \rangle = \sigma^2 \delta_{jk} \delta(t - t').$$

- noise is high dimensional ($128 \times N_{amp}$)
- Amplified spontaneous emission (**ASE**) **noise** produces **jitter** in soliton parameters (A, T, Ω, Φ)

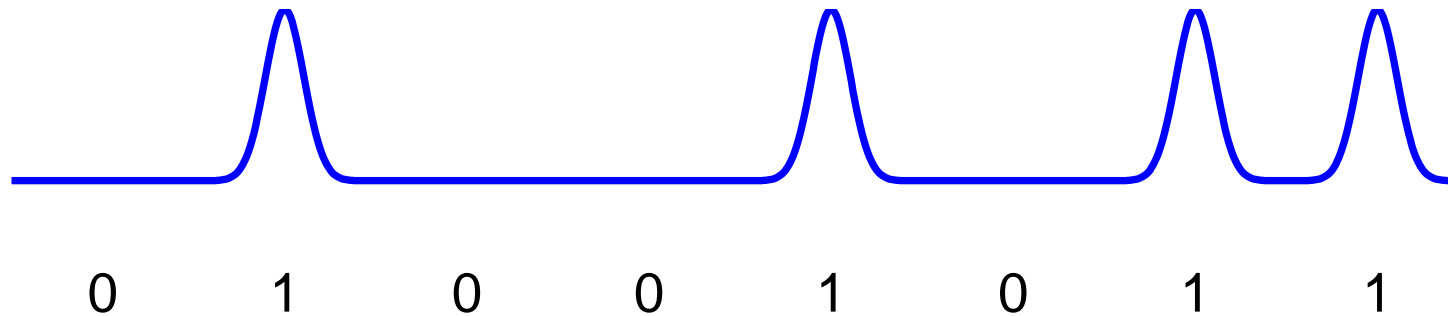
Amplification and noise

The light signal, u , is attenuated and amplified periodically



- Amplifier noise is approximately Gaussian, but when coupled with nonlinearity resulting statistics may not be
- Strong noise-induced signal distortion results in errors

Amplitude-shift keying (ASK)

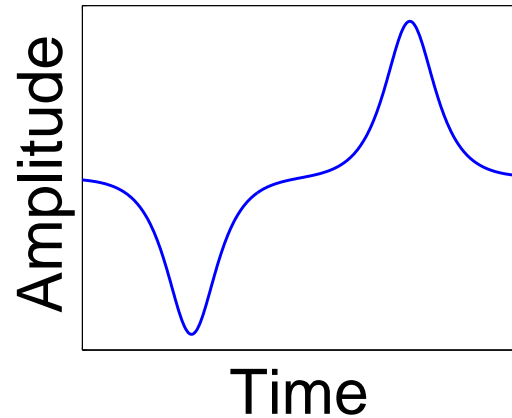


- logical one '1' represented by a pulse of light
- logical zero '0' represented by space (no optical energy)

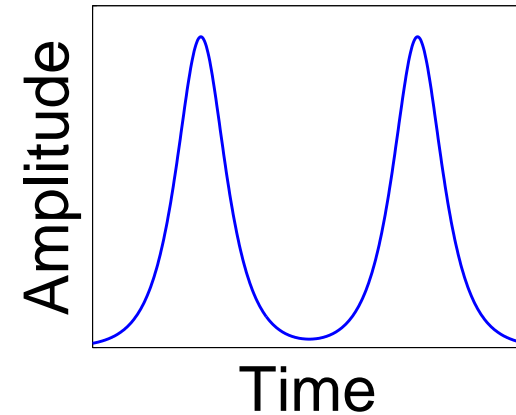
What do we detect at the receiver?

Pulse energy (note $E = E(A)$)

Differential **phase-shift** keying (DPSK)



π phase difference \rightarrow '0'



no phase difference \rightarrow '1'

- encode information on relative phase of adjacent pulses
- more robust than ASK when faced with noise and power fluctuations

But signal is amplified to compensate for loss \Rightarrow amplification noise

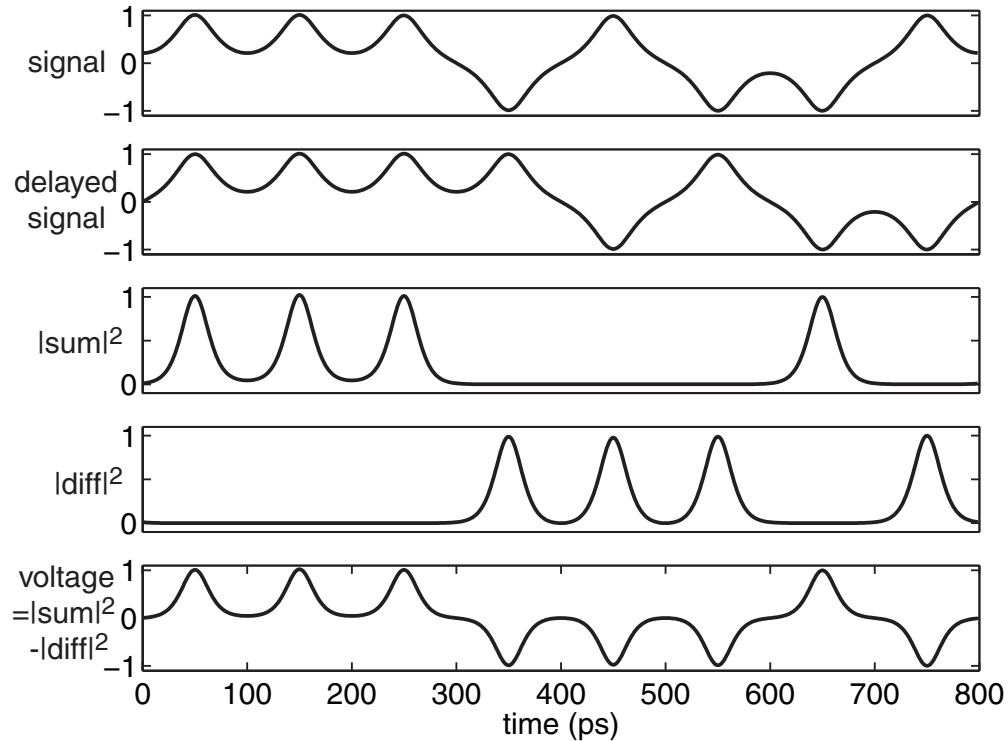
how does noise + propagation change '0' to '1' (or vise-versa)?

How do we decode the phase information?

Optical signal at
detector of bit pattern
11100010

Detector:

$$V = A_1^* A_2^* \cos(\varphi_1^* - \varphi_2^*)$$



To decode: delay one bit period, interfere constructively and destructively, take intensity of each, take difference

Errors: '1' sent but '0' detected or vice-versa

Threshold between a '1' and '0' is **zero voltage**

Noise-induced errors

Predicting error rates:

- Error rates must be small; e.g., 10^{-9} or 10^{-12}
- Simulating rare events with **Monte Carlo** is prohibitively expensive

Understanding why errors occur:

- What combination of impairments produces errors?
- If one knows how errors arise, they can potentially be corrected

One way to see errors: use **importance sampling** to bias simulations toward these events

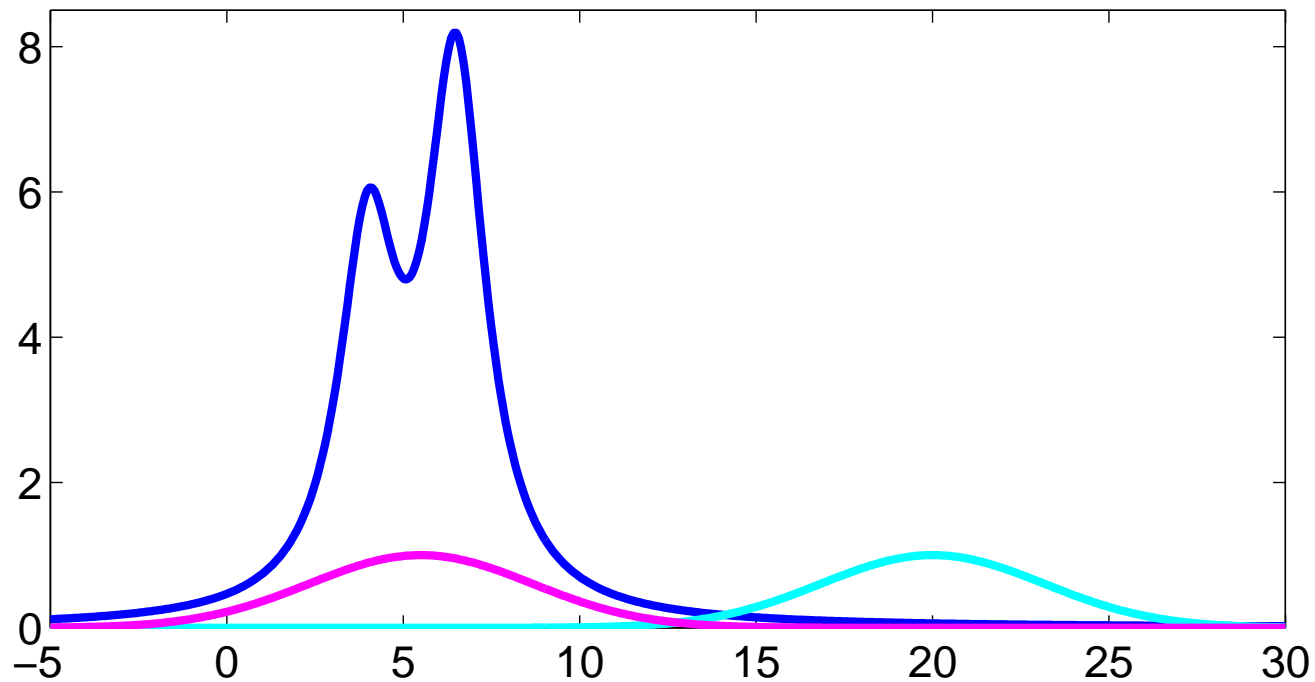
- **Crucial to understand most likely noise realizations that lead to desired events**

Perturbation theory to study noise-induced signal degradation

Aside on MC simulation

- Calculating a probability amounts to calculating an integral
- Numerical integration methods: trapezoid, Simpson, MC, ...

$$\int_{-\infty}^{\infty} f(x) dx = \frac{1}{M} \sum_{k=1}^M f(x_k) \quad (1)$$



Perturbation theory for NLS solitons

- NLS+generic perturbation:

$$iu_z + \frac{1}{2} u_{tt} + |u|^2 u = N(t, z).$$

- Look for perturbative solution:

$$u(t, z) = u_s(t, z) + v(t, z),$$

with $u_s =$ solution of the unperturbed problem

$$u_s = A \operatorname{sech}(A[t - T - \Omega z]) \exp(i[\Omega t + \frac{1}{2}(A^2 - \Omega^2)z + \phi]),$$

- Then $v(t, z)$ solves the *linearized NLS equation*:

$$L v = N(t, z),$$

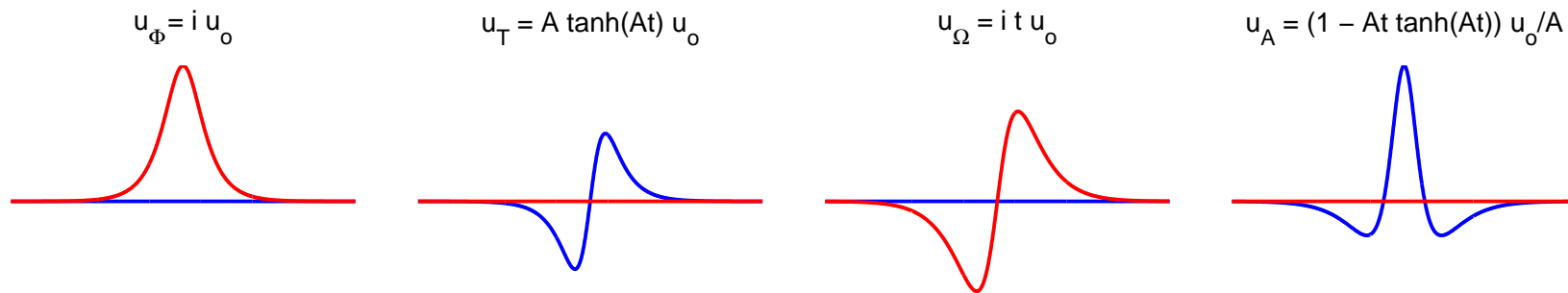
with $L =$ linearized NLS operator:

$$L v = iv_z + \frac{1}{2} v_{tt} + 2|u_s|^2 v + u_s^2 v^*.$$

Linearized NLS operator

- L is non-self-adjoint linear operator

$U_\Phi, U_T, U_A, U_\Omega$ eigenmodes and generalized eigenmodes.



key points

- The linear modes are associated to invariances of the NLS equation
- They each correspond to changes in the soliton parameters
- Each value of the soliton parameters yields an unperturbed solution, so the system does not resist perturbations along these directions.

Soliton perturbation theory cont.

$$u_s = A \operatorname{sech}(A[t - T - \Omega z]) \exp(i[\Omega t + \frac{1}{2}(A^2 - \Omega^2)z + \phi])$$

- Invariances of the NLS equation suggest that it cannot resist **parameter jitter**
- Build up of parameter changes \Rightarrow **large pulse distortion**
- perturbation induced change to a soliton solution, $u = u_s + v$, given by
$$v = \Delta A u_A + \Delta \Omega u_\Omega + \Delta T u_T + \Delta \phi u_\phi + R(t, z)$$
where u_K are **modes of linearized NLS** corresponding to parameter K

Linear modes link perturbations to parameter changes

Importance sampling: a very simple example

Experiment: 100 coin flips

Question: What is the probability of 70 or more heads?

Answer: 2.4×10^{-13}

But, how do we simulate this directly?

Solution: **Use an unfair coin!**

Optimal: Use a coin that gives heads 70% of the time.

Correct for **biased coin** by using **likelihood ratio**:

If on a flip one gets heads, multiply by $0.5/0.7$

If on a flip one gets tails, multiply by $0.5/0.3$

This corrects statistics: get results for a *fair coin*

But, 10 orders of magnitude simulation speedup

IS application: calculating rare events in nonlinear communication systems

goals: predict error rates and understand why errors occur

recall: error rates must be small $< 10^{-10}$

know: how noise changes soliton parameters \Rightarrow **linear modes**

idea: use linear modes to bias simulations toward rare events of interest

but:

Crucial to understand most likely noise realizations that lead to desired events, need to be careful

Optimal biasing for DMNLS

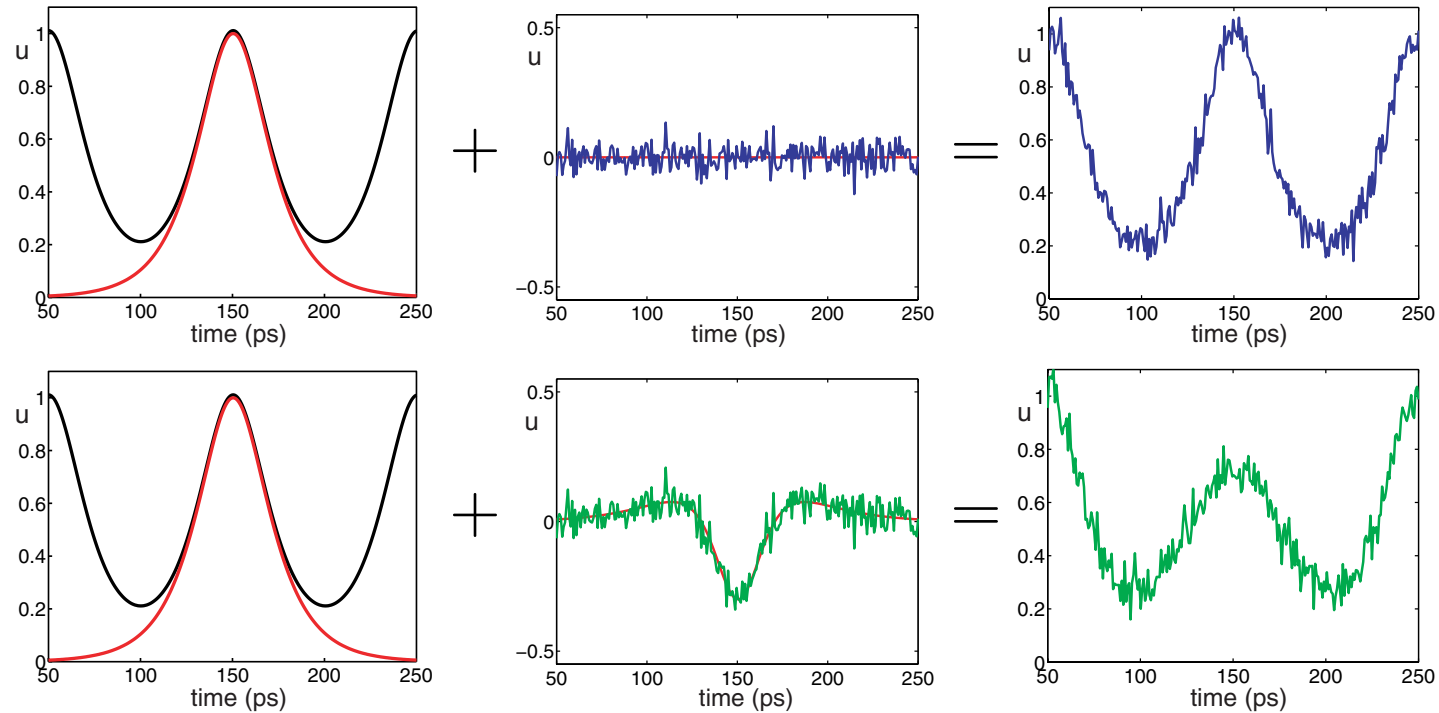
- Consider a noise-induced perturbation, v , to the solution u_S

$$v = \Delta T u_T + \Delta \Omega u_\Omega + \Delta \phi u_\phi + \Delta \lambda u_\lambda + R$$

- Bias noise with function of time $b(t)$
- $n(t)$ white, Gaussian i.i.d., $v(t) = n(t) + b(t)$
- Maximize probability of hitting, on average, a desired parameter change, ΔK , ($K \in \{T, \Omega, \phi, \lambda\}$)
- Solution: $b(t) \propto \Delta K \underline{u}_K(t)$

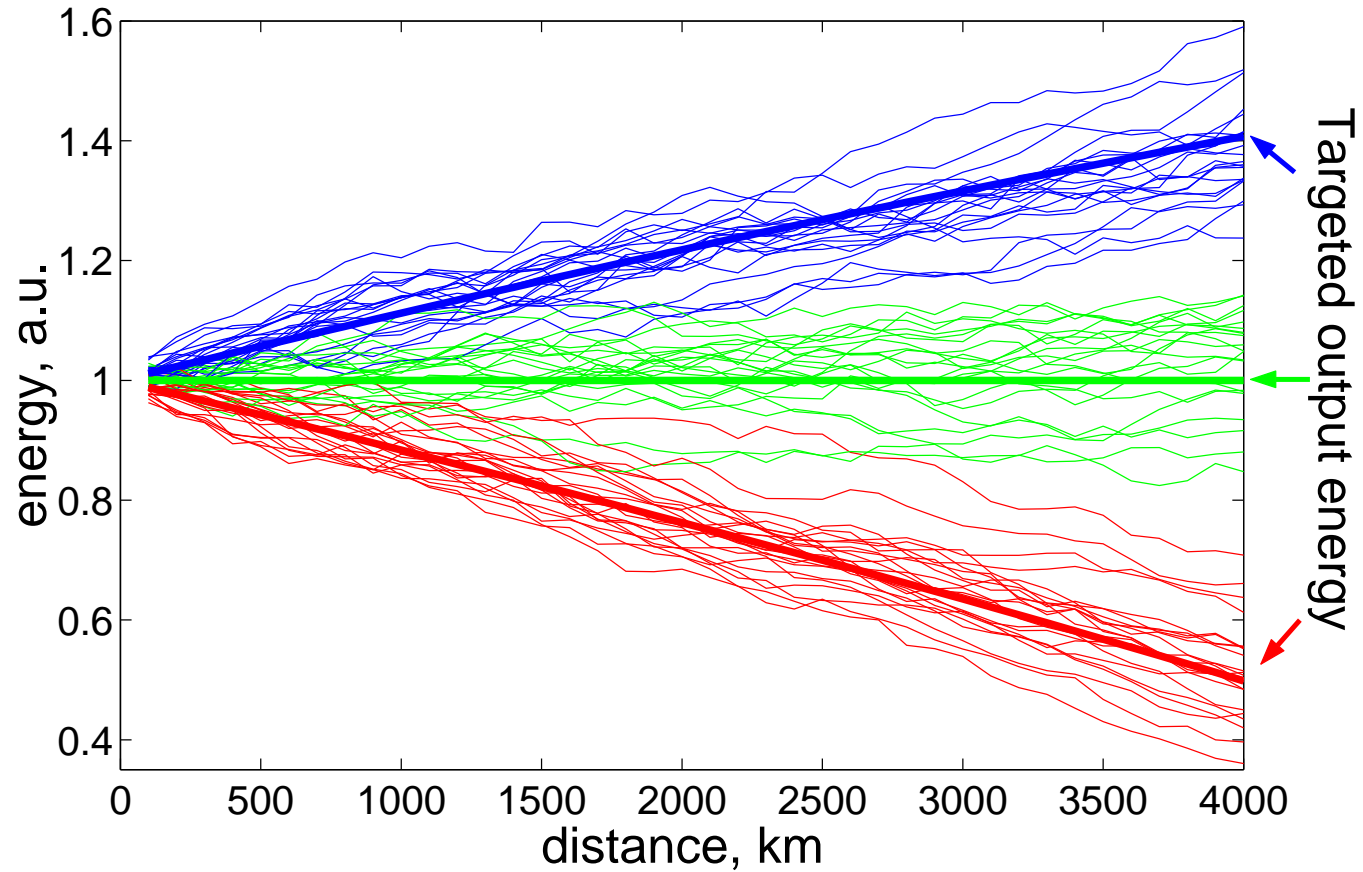
Importance-sampled Monte-Carlo for NLS

Pulse **without** and **with** biasing by the amplitude mode



- correct with likelihood ratio \Rightarrow unbiased statistics of rare events
- how *much* should be bias noise at each amplifier?

Samples following targeted path

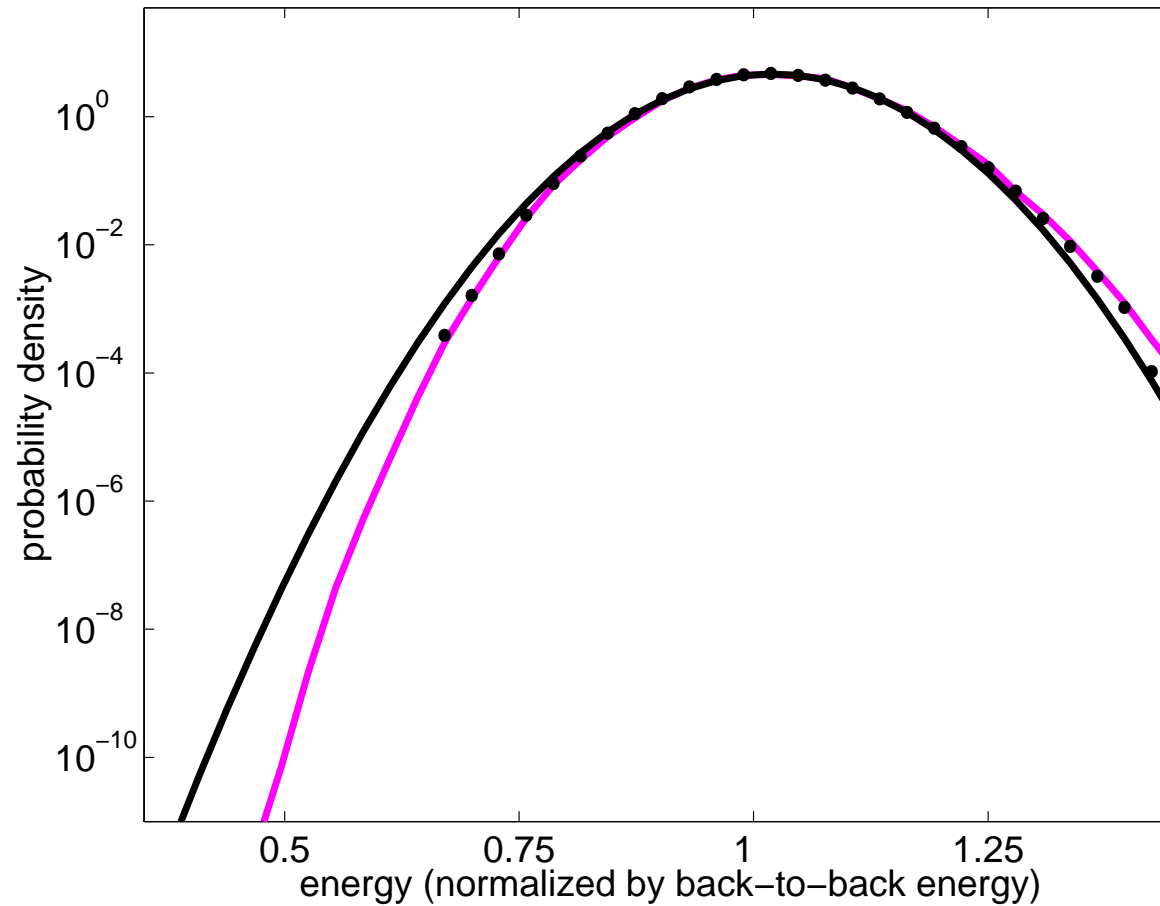


- correct statistics with likelihood ratio \Rightarrow unbiased statistics of rare events

Importance sampled Monte-Carlo algorithm

1. Generate a signal
2. Propagate signal to next amplifier (split-step Fourier)
3. Fit soliton to noisy signal
4. Using analytic formulas, compute eigenmodes of the NLS equation linearized about the soliton
5. Generate zero-mean normal random variables
6. Bias r.v. with the optimal combo of linearized modes
7. Update the likelihood ratio, $p(\mathbf{X})/p^*(\mathbf{X})$
8. Repeat steps 2-7 until end of transmission line
9. Detect signal
10. Update appropriate bin with likelihood ratio

IS-MC results: energy pdf



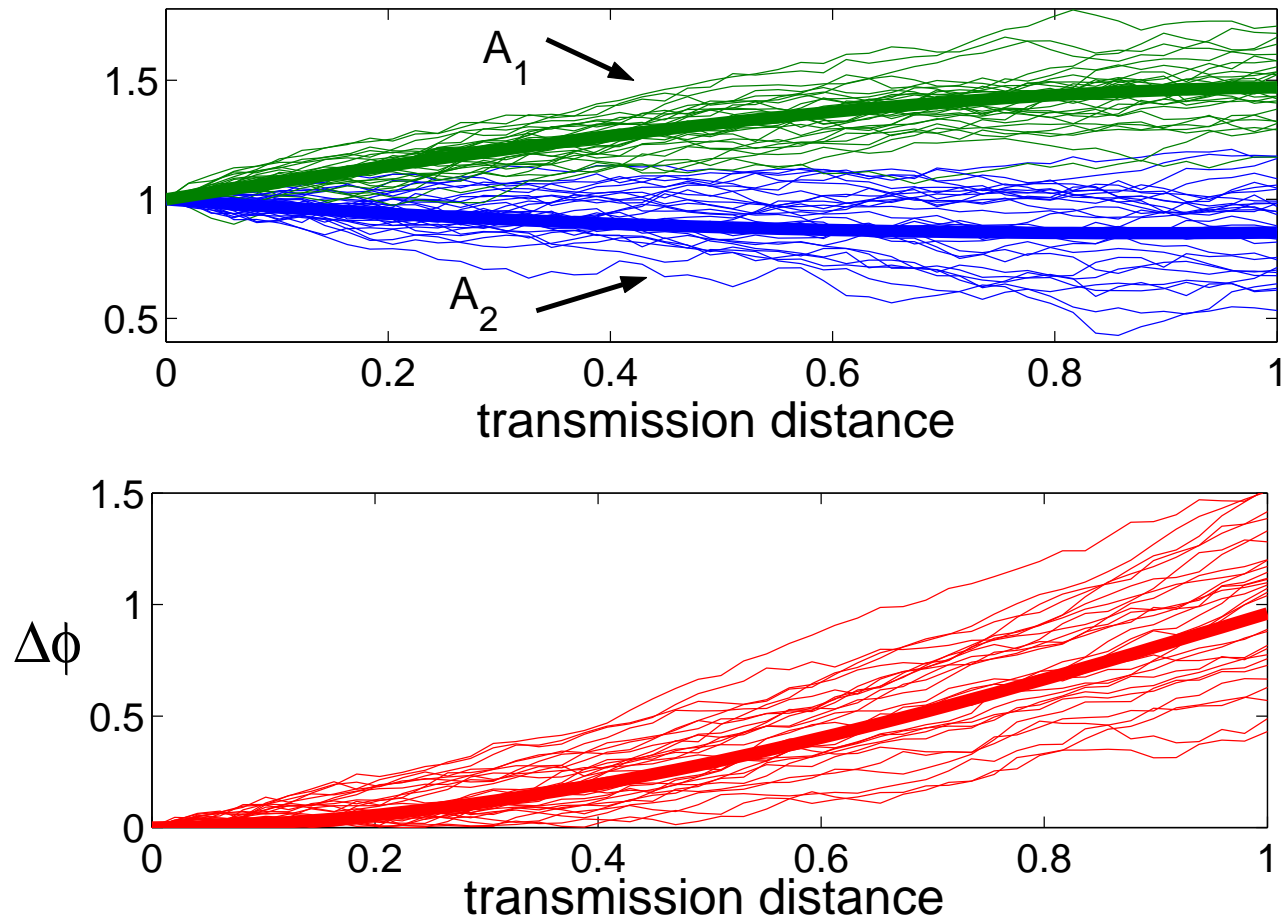
black dots - 50,000 MC samples

black curve - Gaussian fit to MC simulations

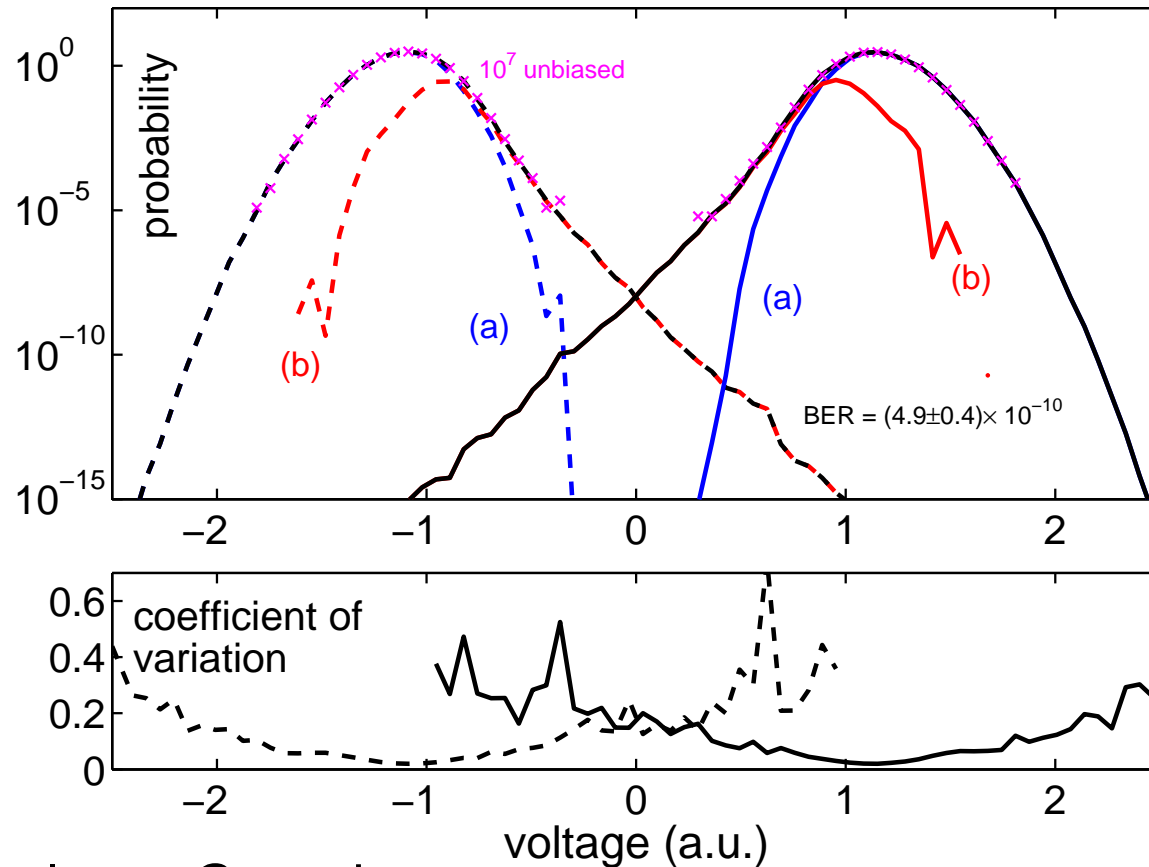
pink curve - 42,000 IS-MC samples (DMNLS)

Typical biased parameter paths

Parameter paths from a few biased trials – anti-correlated



Full simulation: voltage PDFs for DPSK



- PDFs clearly not Gaussian.
- **Blue** and **red** curves are contributions to the overall PDFs from **correlated (large prob)** and **anticorrelated (small prob)** events.

Rare events in nonlinear lightwave systems

Conclusions

- Demonstration of importance sampled Monte-Carlo simulations of rare events in soliton-based optical communication systems
- PDFs can be sampled way down into the tails
- Perturbation theory to guide importance sampling
- Examples: pdfs of energy and voltage
- This method can be used as a practical tool to analyze the performance of nonlinear optical systems