

Introduction to the Forward Problem: Solving the Harmonic Oscillator

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Introduction

$$\sum \mathbf{F} = m\mathbf{a}$$

Consider a mass on a spring sitting on the table. What are the forces acting on the object?

- ▶ Gravity
- ▶ The spring
- ▶ Friction

We only care about one direction we will call y . So ignoring gravity we have

$$m\mathbf{a} = \mathbf{F}_s + \mathbf{F}_f$$

Let's add a third force, a forcing input.

$$m\mathbf{a} = \mathbf{F}_s + \mathbf{F}_f + \mathbf{F}_I.$$

If $\ddot{y} = \mathbf{a}$ and $\dot{y} = \mathbf{v}$, then we can write

$$m\ddot{y} = ky + b\dot{y} + \mathbf{F}_I.$$

The force in the spring depends on how much you stretch or compress it. The force due to friction depends on how fast things are moving.

Differential equation representation

Without any loss of generality or other problems we can write

$$m\ddot{y} + b\dot{y} + ky = \mathbf{F}(t).$$

This is a second-order differential equation. Naturally we want to solve for $y(t)$. In order to do this let's first think of a simpler case.

$$\begin{aligned}\ddot{y} - y &= 0 \text{ where } b = 0, \text{ and } k = -m \\ \ddot{y} &= y\end{aligned}$$

The solution here is either $y(t) = e^t$ or $y(t) = e^{-t}$. In order to solve for this completely we need to know $y(0)$ and $\dot{y}(0)$, the initial conditions. The solution will then have the form

$$c_1 e^t + c_2 e^{-t}$$

Homogeneous solution based on the characteristic equation

Intuitively speaking, we might think that if we knew the initial conditions $y(0)$ and $\dot{y}(0)$, the solution to a problem of the form

$$a\ddot{y} + b\dot{y} + cy = 0$$

might be related to the previous solution. In order to explore this we note that if $y = e^{rt}$ then $\dot{y} = re^{rt}$ and $\ddot{y} = r^2e^{rt}$. Then we can write

$$\begin{aligned} a\ddot{y} + b\dot{y} + cy &= 0 \\ (ar^2 + br + c)e^{rt} &= 0 \\ ar^2 + br + c &= 0 \end{aligned}$$

If we see a second order polynomial we should always find the roots.

If we do this we will see that the solution is then

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

where r_1 and r_2 are the roots of the characteristic equation. Once again given the initial conditions we can solve for c_1 and c_2 .

Let's return to our simple case. Assume that there is no friction and no input, then

$$m\ddot{y} + ky = 0$$

$$\ddot{y} + \frac{k}{m}y = 0$$

$$(r^2 + \frac{k}{m})e^t = 0$$

$$(r^2 + \frac{k}{m}) = 0$$

$$r = \pm\sqrt{-\frac{k}{m}}$$

Complex Roots

So, our solution is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

But, there is a problem the roots are imaginary. It turns out through the identity

$$e^{it} = \cos t + i \sin t$$

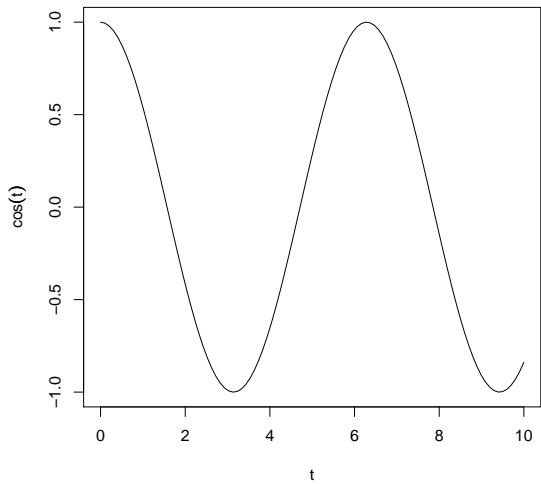
and some vigorous hand-waving, we can show that the solution is

$$y(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}t\right).$$

This further reduces to

$$y(t) = A \cos(\omega_0 t + \phi)$$

where the constants A and ϕ are determined by the initial conditions, and $\omega_0 = \sqrt{k/m}$.



Adding Damping

This solution actually makes perfect sense. In the absence of friction we would expect that the block would just oscillate back and forth with the amplitude of the oscillation determined by how far we stretched the spring and the phase determined by the time we released the block, and the frequency determined by the ratio of the spring's stiffness to the mass of the block.

What about the case with friction (damping)?

In that case then we solve the more complex problem and we find the roots of the characteristic equation

$$ar^2 + br + c = 0$$

In this case when we consider the friction or differential equation is

$$\ddot{y} + \frac{b}{m}\dot{y} + \frac{k}{m}y = 0$$

So, in this case $a = 1$, $b = b/m$ and $c = k/m$. We will also find in this case that our roots are complex conjugates of the form

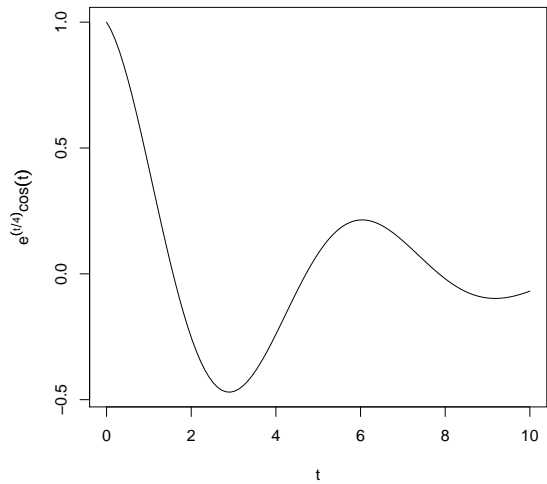
$$\lambda \pm i\mu$$

That leads to a solution of the form

$$y(t) = Ae^{\lambda t} \cos(\omega_0 t + \phi)$$

where the quantities λ and ω_0 are determined by the physical characteristics of our system and the terms A and ϕ are determined by the initial conditions.

Typically $\lambda < 0$, and then the solution makes perfect sense, the oscillations die down eventually at a rate determined by λ , which depends on the mass m and the damping coefficient b .



Forced vibrations

So, up to this point the solution is pretty straight-forward. But what about the case where we have a forcing frequency? What if we shake the block?

In this case our differential equation is then

$$\ddot{y} + \frac{b}{m}\dot{y} + \frac{k}{m}y = F$$

where F is some forcing input, let's assume that $F = F_0 \cos(\omega t)$

Again consider the simple model without damping

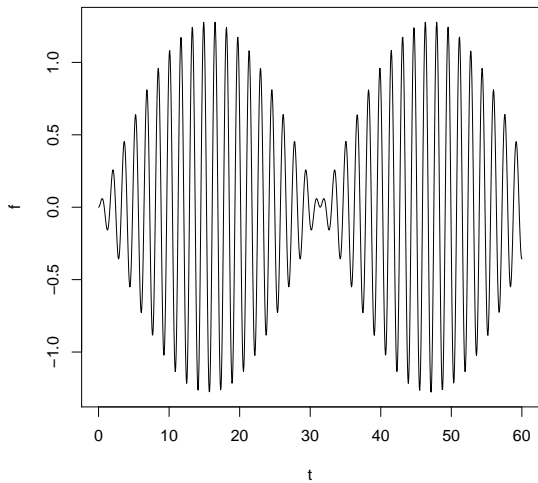
$$\ddot{y} + \frac{k}{m}y = F_0 \cos(\omega t), \text{ where } \sqrt{k/m} \neq \omega$$

In this case the solution will look like this

$$y(t) = A \cos(\omega_0 t + \phi) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t).$$

If $\phi = 0$ then the solution is

$$y(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t - \omega_0 t)$$



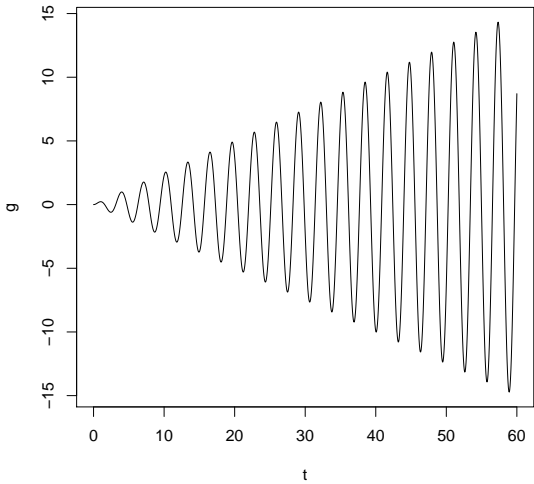
Resonant forcing frequencies

What about the case where $\omega_0 = \omega$?

In this case the solution then becomes

$$y(t) = \cos(\omega_0 t) \frac{F_0}{2m\omega_0} t \sin(\omega_0 t)$$

In reality there is always some form of damping or some limitation on the magnitude of $y(t)$, but in practice this use of a resonant forcing frequency can be useful.



What does this have to do with the beam problem?

Well the beam does behave like a spring. The vibrations of the beam are in fact pretty well modeled as a second order differential equation.

The State-Space Model

In order to get MATLAB to solve a second order differential equation, it needs to be written as a set of two first order differential equations. This is actually quite simple. Given our equation

$$\ddot{y} + \frac{b}{m}\dot{y} + \frac{k}{m}y = 0.$$

Let

$$\begin{aligned}z_1 &= y(t) \\z_2 &= \dot{y}\end{aligned}$$

then

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= -\frac{k}{m}z_1 - \frac{b}{m}z_2\end{aligned}$$

In matrix form then this can be written as

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

with the initial conditions

$$\mathbf{z}_0 = \begin{bmatrix} y_0 \\ \dot{y}_0 \end{bmatrix}$$

which is in a form that MATLAB can handle.

The Inverse Problem

What if we didn't know m , b or k ?

This is the inverse problem, given some observed data can we find out the values for those parameters?