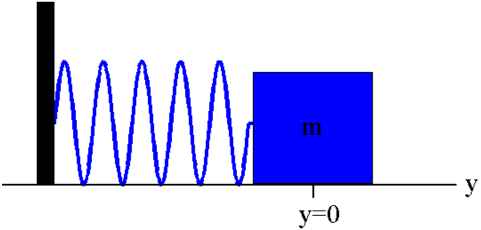
Introduction to the Forward Problem: the Harmonic Oscillator

SAMSI Undergraduate Workshop 2008

May 19, 2008

Spring-mass system



Newton's Second Law $$\sum \boldsymbol{F} = m\boldsymbol{a}$$

What are the forces acting on the object?

- The spring
- Friction

Newton's Second Law (1-D)

Motion is in one direction, y is the displacement of the mass

$$\boldsymbol{v} = \dot{y} \quad \text{and} \quad \boldsymbol{a} = \ddot{y} \quad (\text{where } \dot{y} = \frac{dy}{dt})$$

- The spring force depends on how much the spring is stretched or compressed

$$\boldsymbol{F}_{spring} = -ky$$

- The friction force depends on how fast things are moving

$$\boldsymbol{F}_{friction} = -c\dot{y}$$

From $\boldsymbol{F} = m\boldsymbol{a}$ to ODE

In terms of displacement y, $\boldsymbol{F}_{spring} + \boldsymbol{F}_{friction} = m\boldsymbol{a}$ becomes

$$m\ddot{y} = -c\dot{y} - ky \text{ (+maybe some } \textit{external} \text{ forces)}$$

Without external forces the displacement is described by

$$m\ddot{y} + c\dot{y} + ky = 0$$

This is a second-order homogeneous differential equation.

<u>Question</u>
How does $y(t)$ behave?

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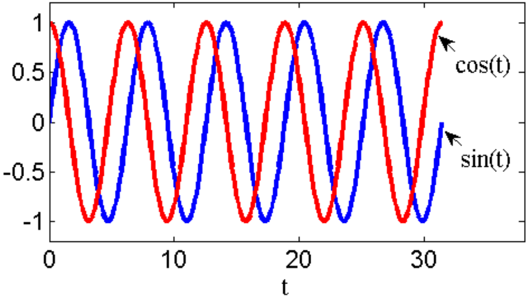
Let's first think of a simpler case: no friction ($c = 0, k = m$)

$$\ddot{y} + y = 0 \quad \text{i.e.,} \quad \ddot{y} = -y$$

Question

What function(s) is its own second derivative?

Sine and Cosine



$$y(t) = c_1 \sin(t) + c_2 \cos(t)$$

In order to solve we need to know initial conditions $y(0)$ & $\dot{y}(0)$

How does $y(t)$ behave?

Another simpler case: no spring ($k = 0, c = m$)

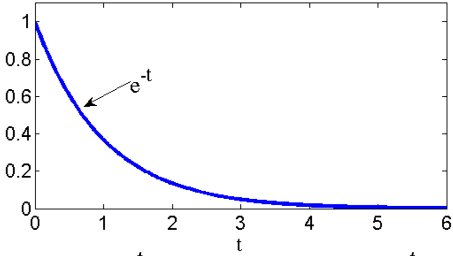
$$\ddot{y} = -\dot{y} \quad \text{or} \quad \dot{v} = -v$$

Question

What function is (the negative of) *its own derivative?*

Exponential

$$\dot{v} + v = 0 \quad \text{(and remember, } \dot{y} = v\text{)}$$



$$v(t) = c_1 e^{-t} \quad \text{and} \quad y(t) = -c_1 e^{-t} + c_2$$

Question

If we put the two simpler cases together, what happens?

Homogeneous solution: characteristic equation

Goal We seek the solution of an ODE with the form

$$(*) \quad A\ddot{y} + B\dot{y} + Cy = 0$$

Know Should be related to the previous (simple) solutions.

Try Let's guess a solution with the form $y = e^{rt}$. Then we have

$$\begin{aligned} A\ddot{y} + B\dot{y} + Cy &= 0 \\ (Ar^2 + Br + C)e^{rt} &= 0 \\ (**) \quad Ar^2 + Br + C &= 0 \end{aligned}$$

Note If r solves **, $y(t) = e^{rt}$ solves *

Let's try this for our equation $m\ddot{y} + c\dot{y} + ky = 0$

Plug in $y = e^{rt}$ and see $r^2 + \frac{c}{m}r + \frac{k}{m} = 0$

which has solutions

$$r_1 = -\frac{c}{2m} + \sqrt{(\frac{c}{2m})^2 - \frac{k}{m}} \quad \text{and} \quad r_2 = -\frac{c}{2m} - \sqrt{(\frac{c}{2m})^2 - \frac{k}{m}}$$

we see that the solution to our differential equation is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Question

Where are the sines & cosines?

Underdamped system $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$$r_1 = -\frac{c}{2m} + \sqrt{(\frac{c}{2m})^2 - \frac{k}{m}} \quad \text{and} \quad r_2 = -\frac{c}{2m} - \sqrt{(\frac{c}{2m})^2 - \frac{k}{m}}$$

Really we have three cases:

1. $(\frac{c}{2m})^2 - \frac{k}{m} > 0$ overdamped
2. $(\frac{c}{2m})^2 - \frac{k}{m} = 0$ critically damped
3. $(\frac{c}{2m})^2 - \frac{k}{m} < 0$ underdamped

Question

What happens if $\frac{k}{m} > (\frac{c}{2m})^2$?

Complex roots

$$(\frac{c}{2m})^2 - \frac{k}{m} < 0 \qquad \text{underdamped (complex root)}$$

Recall $\qquad i = \sqrt{-1}$

$$\sqrt{-1(\frac{k}{m} - (\frac{c}{2m})^2)} = i\sqrt{\frac{k}{m} - (\frac{c}{2m})^2} = i\omega \text{ where } \omega^2 = \frac{k}{m} - (\frac{c}{2m})^2$$

so we can write our roots as

$$r_1 = -\frac{c}{2m} + i\sqrt{\omega} \quad \text{and} \quad r_2 = -\frac{c}{2m} - i\sqrt{\omega}$$

But *Where are the sines and cosines?*

Euler's Equation and the underdamped solution

Recall our solution is $y(t) = c_1 e^{-\frac{c}{2m}t + i\omega t} + c_2 e^{-\frac{c}{2m}t - i\omega t}$

and Euler's equation says $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$

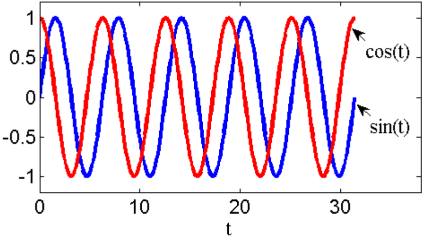
so we can write our solution as

$$y(t) = e^{-\frac{c}{2m}t}(C_1 \cos(\omega t) + C_2 \sin(\omega t))$$

... but it's a little easier to think about if we rewrite it again

Keep in mind *Sine is the same as Cosine with a phase shift*

In case you're not convinced

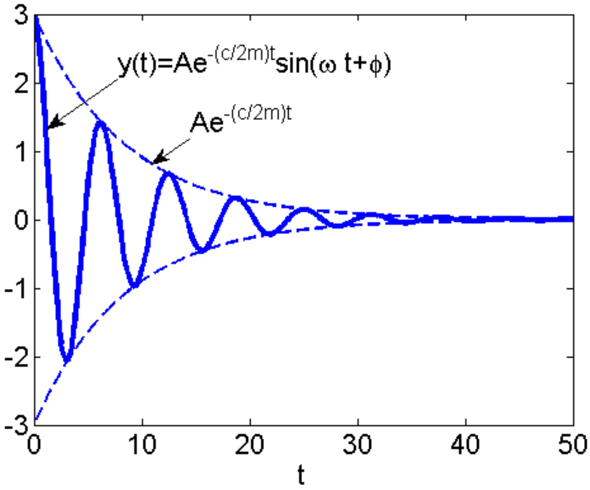


$$y(t) = e^{-\frac{c}{2m}t}(C_1 \cos(\omega t) + C_2 \sin(\omega t))$$

$$\Downarrow$$ (messy algebra)

$$y(t) = Ae^{-\frac{c}{2m}t} \sin(\omega t + \phi)$$

Underdamped spring-mass behavior



Forced vibrations

What about the case where we have a forcing frequency?

What if we shake the block?

In this case our differential equation is then

$$\ddot{y} + \frac{c}{m}\dot{y} + \frac{k}{m}y = F$$

where F is some external forcing, assume that $F = F_0 \cos(\alpha t)$

Nonhomogeneous equation

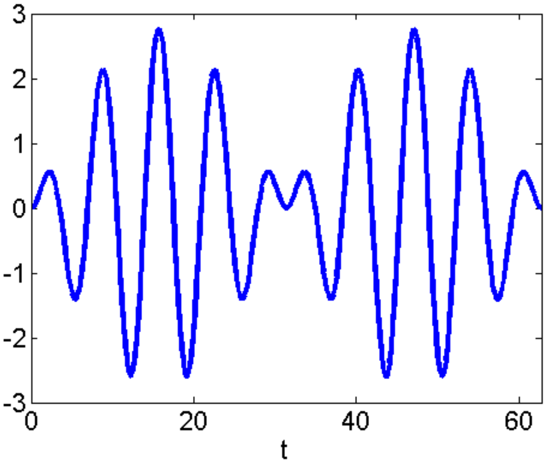
Again consider a simpler model without damping ($c = 0,\ \&\ \omega^2 = k/m$)

$$\ddot{y} + \omega^2 y = F_0 \cos(\alpha t),\ \text{where } \omega \neq \alpha$$

In this case the solution will look like this

$$y(t) = A \sin(\omega t + \phi) + \frac{F_0}{m(\omega^2 - \alpha^2)} \cos(\alpha t).$$

Forced behavior



Resonant forcing frequencies

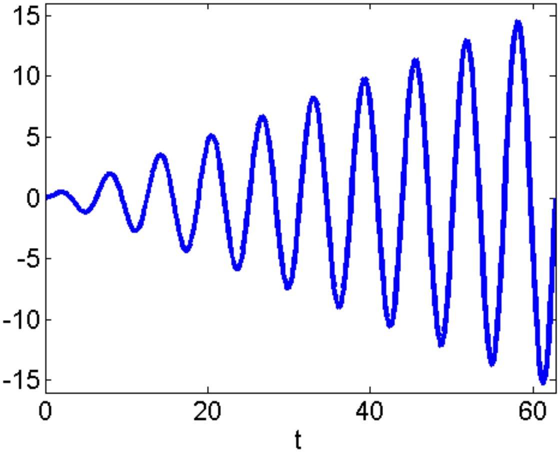
What about the case where $\alpha = \omega$?

That is, what if we force at the resonant frequency?

In this case the solution then becomes

$$y(t) = A\sin(\omega t + \phi) + \frac{F_0}{2m\omega} t \sin(\omega t)$$

Secular growth



Damped secular growth

In reality there is always some form of damping or some limitation on the magnitude of $y(t)$, but in practice this use of a resonant forcing frequency can be useful.

Question   When have you used resonant forcing?

Why are we thinking about a spring-mass?

What does this have to do with the beam problem?

- The beam more or less behave like a spring.
- The vibrations of the beam are pretty well modeled by a second order differential equation.

Second-order equation as first-order system

For MATLAB to solve a second order differential equation, it needs to be written as a set of two first order ODEs.

Given our equation $\ddot{y} + \frac{c}{m}\dot{y} + \frac{k}{m}y = 0.$

Let

$$\begin{aligned} z_1 &= y(t) \\ z_2 &= \dot{y} \end{aligned}$$

and let $C \equiv \frac{c}{m}$ and $K \equiv \frac{k}{m}$ then we see

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -Kz_1 - Cz_2 \end{aligned}$$

Linear system

In matrix form then this can be written as

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K & -C \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

or

$$\dot{\boldsymbol{z}} = \mathbf{A}\boldsymbol{z}$$

with the initial conditions

$$\boldsymbol{z}_0 = \begin{bmatrix} y_0 \\ \dot{y}_0 \end{bmatrix}$$

which is in a form that MATLAB can handle.

The Inverse Problem

What if we don't know C or K?

Challenge

Given some observed data (or collecting it yourself, as the case may be) & the model we just derived, can we figure them out?