MODEL DEVELOPMENT FOR THIN BEAMS

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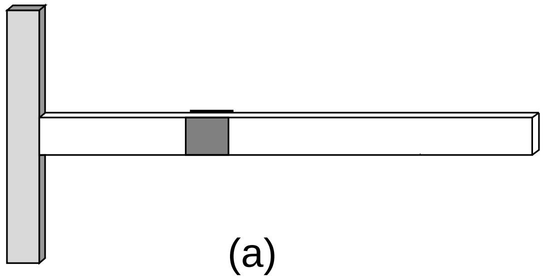
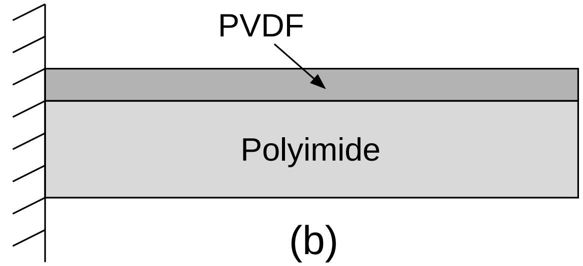
APPLICATIONS

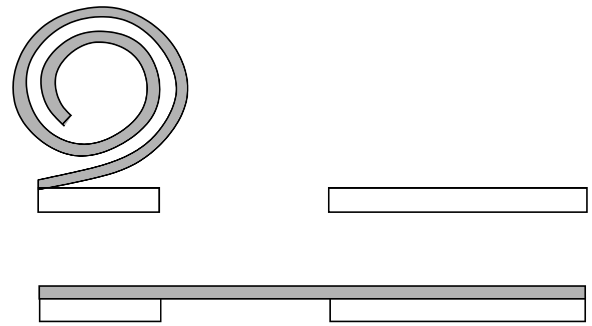


(a)



(b)

(c)



(d)

Note:

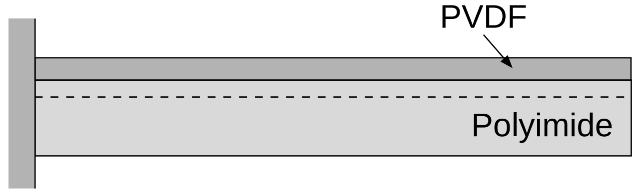
(a) Thin beam with surface-mounted PZT patches employed as a prototype for vibration control.

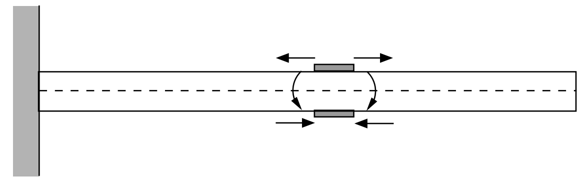
(b) Polymer unimorph comprised of PVDF and polyimide presently considered for pressure sensing and flow control.

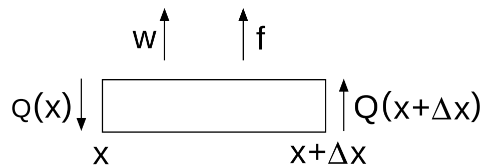
(c) Curved THUNDER transducer whose width is small compared with the length.

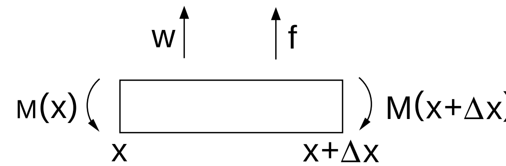
d) Electrostrictive MEMs actuator employed as a high speed shutter.

FORCE AND MOMENT BALANCING









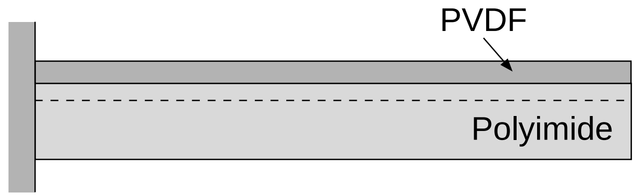
Force Balance:

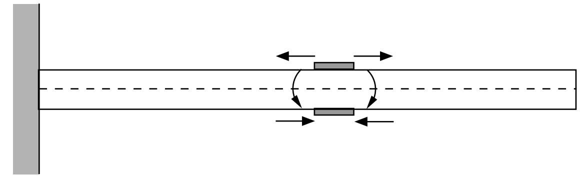
$$\int_x^{x+\Delta x} \rho \frac{\partial^2 w}{\partial t^2}(t,s)ds = Q(t,x+\Delta x) - Q(t,x) + \int_x^{x+\Delta x} \left[f(t,s) - \gamma \frac{\partial w}{\partial t}(t,s) \right] ds$$

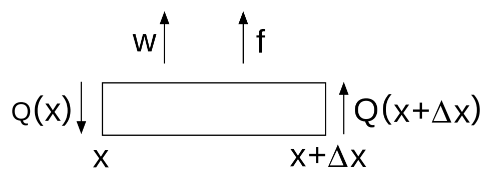
- Divide by Δx and take $\Delta x \to 0$ to obtain

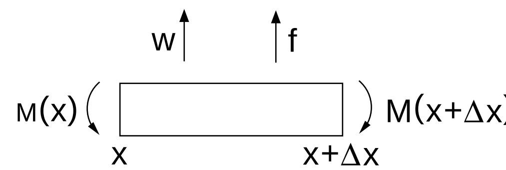
$$\rho \frac{\partial^2 w}{\partial t^2} + \gamma \frac{\partial w}{\partial t} = \frac{\partial Q}{\partial x} + f$$

FORCE AND MOMENT BALANCING









Force Balance:

$$\int_x^{x+\Delta x} \rho \frac{\partial^2 w}{\partial t^2}(t,s)ds = Q(t,x+\Delta x) - Q(t,x) + \int_x^{x+\Delta x} \left[f(t,s) - \gamma \frac{\partial w}{\partial t}(t,s) \right] ds$$

- Divide by Δx and take $\Delta x \rightarrow 0$ to obtain

$$\rho \frac{\partial^2 w}{\partial t^2} + \gamma \frac{\partial w}{\partial t} = \frac{\partial Q}{\partial x} + f$$

Moment Balance:

$$M(t,x+\Delta x) - M(t,x) - Q(t,x+\Delta x)\Delta x + \int_x^{x+\Delta x} f(t,s)(s-x)dx = 0$$

- Note: Limit yields $Q = \dfrac{\partial M}{\partial x}$

- Beam Model: $\rho \dfrac{\partial^2 w}{\partial t^2} + \gamma \dfrac{\partial w}{\partial t} - \dfrac{\partial^2 M}{\partial x^2} = f$

MOMENT RELATION

Heuristic: $$M(t,x) = -\alpha \frac{\partial^2 w}{\partial x^2}$$

$$\alpha = Y \frac{h^3 b}{12}$$

MOMENT RELATION

Heuristic: $M(t,x) = -\alpha \frac{\partial^2 w}{\partial x^2}$

$$\alpha = Y \frac{h^3 b}{12}$$

More Generally:

$$M(t,x) = -YI(x)\frac{\partial^2 w}{\partial x^2} - cI(x)\frac{\partial^3 w}{\partial x^2 \partial t}$$

BEAM MODEL: STRONG FORMULATION

Strong Formulation:

$$\rho \frac{\partial^2 w}{\partial t^2} + \gamma \frac{\partial w}{\partial t} - \frac{\partial^2 M}{\partial x^2} = f(t,x)$$

$$w(t,0) = \frac{\partial w}{\partial x}(t,0) = 0$$

$$M(t,\ell) = \frac{\partial M}{\partial x}(t,\ell) = 0$$

$$w(0,x) = w_0(x) \quad , \quad \frac{\partial w}{\partial t}(0,x) = w_1(x)$$

where

$$M(t,x) = -YI(x)\frac{\partial^2 w}{\partial x^2} - cI(x)\frac{\partial^3 w}{\partial x^2 \partial t}$$

BEAM MODEL: WEAK FORMULATION

Weak Formulation:

$$\int_0^\ell \rho \frac{\partial^2 w}{\partial t^2}\phi dx + \int_0^\ell \gamma \frac{\partial w}{\partial t}\phi dx - \int_0^\ell M \frac{d^2\phi}{dx^2} dx = \int_0^\ell f\phi dx$$

or

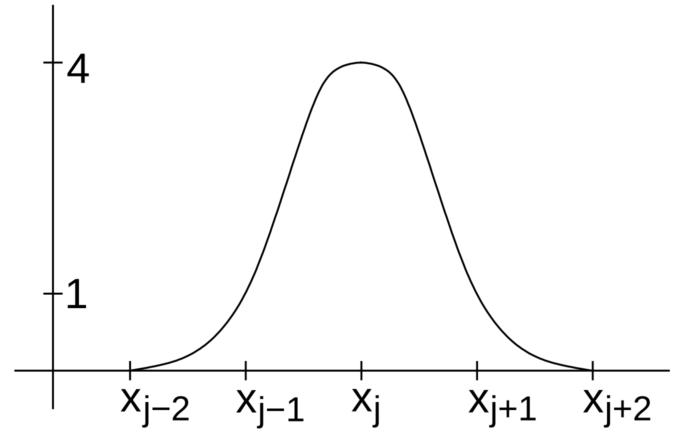
$$\int_0^\ell \rho \frac{\partial^2 w}{\partial t^2}\phi \, dx + \int_0^\ell \gamma \frac{\partial w}{\partial t}\phi \, dx + \int_0^\ell YI \frac{\partial^2 w}{\partial x^2}\frac{d^2\phi}{dx^2} \, dx$$

$$+ \int_0^\ell cI \frac{\partial^3 w}{\partial x^2 \partial t}\frac{d^2\phi}{dx^2} \, dx = \int_0^\ell f\phi \, dx$$

for all appropriate test functions ϕ.

FINITE DIMENSIONAL APPROXIMATION

Basis and Approximate Solution:



- Basis: $\{\phi_j(x)\}$

- Approximate Solution: $w^N(t,x) = \sum_{j=1}^{N+1} w_j(t)\phi_j(x)$

- Second-order Matrix System

$$\mathbb{M}\ddot{\mathbf{w}} + \mathbb{Q}\dot{\mathbf{w}} + \mathbb{K}\mathbf{w} = \mathbf{f}$$

where $\mathbf{w} = [w_1(t)\,,\ \ldots\ ,\ w_{N+1}(t)]^T$ and

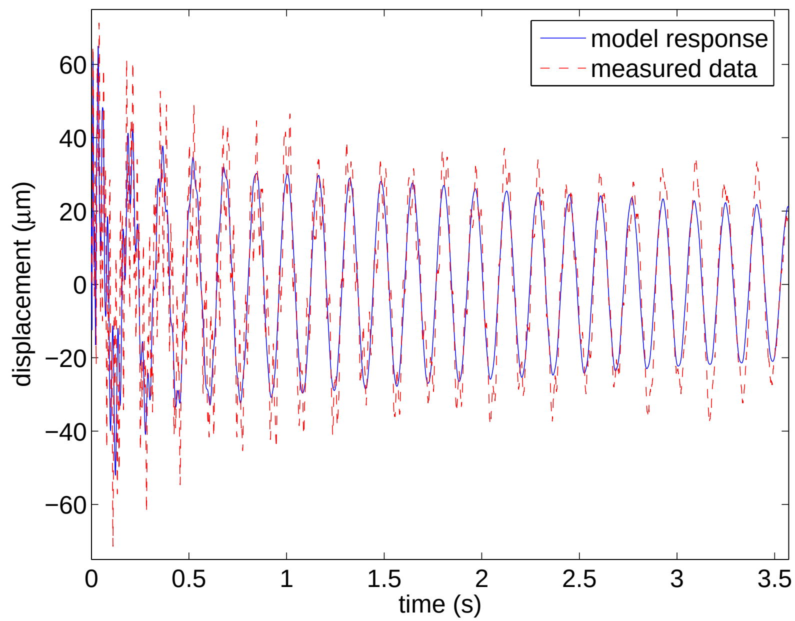
$$[\mathbb{M}]_{ij} = \int_0^\ell \rho\phi_i\phi_j dx$$

$$[\mathbb{Q}]_{ij} = \int_0^\ell \left[\gamma\phi_i\phi_j + cI\phi_i''\phi_j''\right] dx$$

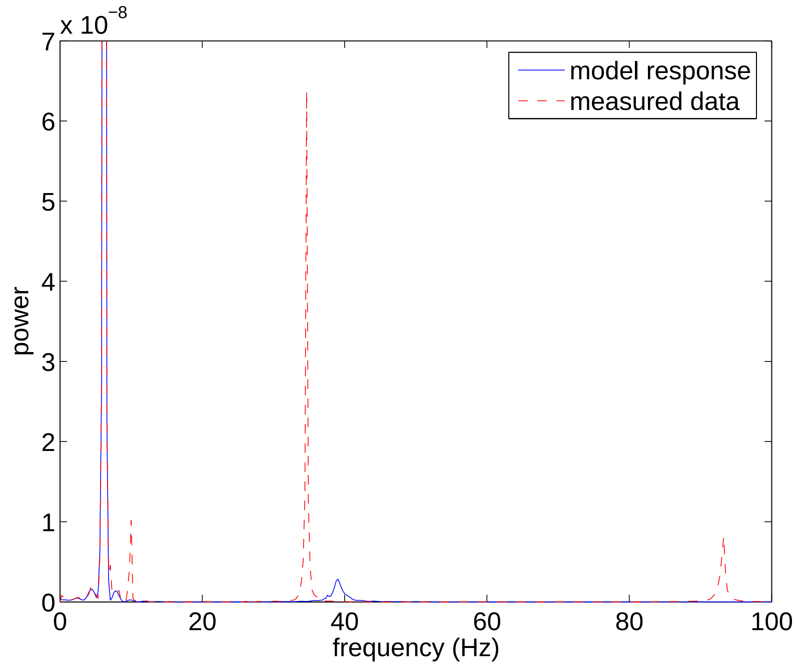
$$[\mathbb{K}]_{ij} = \int_0^\ell YI\phi_i''\phi_j'' dx$$

BEAM APPROXIMATION: CONSTANT PARAMETERS

Time Domain:

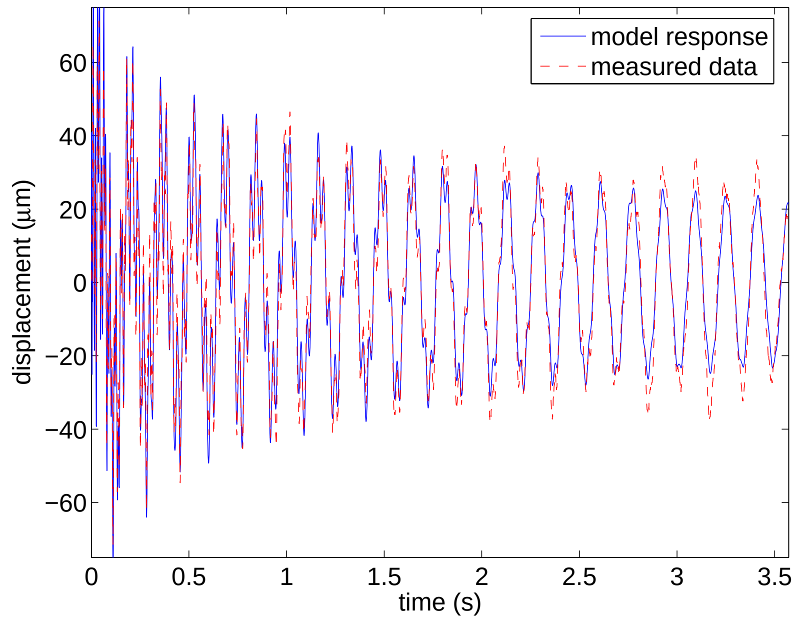


Frequency Domain:



BEAM APPROXIMATION: PIECEWISE-CONSTANT PARAMETERS

Time Domain:



Frequency Domain: